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A SIMPLE PROOF THAT $\zeta(2) = \pi^2/6$

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ABSTRACT. We give a simple proof that $\zeta(2) = \pi^2/6$ using only first-year calculus. An elementary recursion allows us to deduce that generally, $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, a celebrated theorem of Euler.

1. Introduction

In 1650, Pietro Mengoli posed the problem of explicitly evaluating

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and this had come to be known as the Basel problem. The solution is $\pi^2/6$ and it was discovered by Euler almost a century later in 1734, when Euler was 28 years old. Euler's proof, though correct, was far from rigorous and had to await further developments in complex analysis to put it on a sure footing. We will give a slick proof that uses only ideas from first year calculus. In an equally simple way, we will show that $\zeta(2k) \in \pi^{2k}\mathbb{Q}$, where $\zeta(s)$ is the Riemann zeta function which for $s > 1$ is given by

$$\zeta(s) = \sum\nolimits_{n=1}^{\infty} \frac{1}{n^s}.$$

2. The Proof

We first observe that

$$\int_0^\infty \left(\frac{2y}{1+y^2} - \frac{2y}{x^2+y^2} \right) dy = \lim_{N\to\infty} \int_0^N \left(\frac{2y}{1+y^2} - \frac{2y}{x^2+y^2} \right) dy$$
$$= \left[\log \frac{1+y^2}{x^2+y^2} \right]_{y=0}^{y=\infty} = 2\log x, \tag{2.1}$$

since, for fixed x,

$$\lim_{y\to\infty} \log \frac{1+y^2}{x^2+y^2} = 0.$$

Simplifying the integrand in (2.1), we deduce

$$-\frac{\log x}{1-x^2} = \int_0^\infty \frac{y\,dy}{(1+y^2)(x^2+y^2)}. \tag{2.2}$$

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Now consider the integral

$$I = \int_0^1 \frac{\log x}{1-x^2}dx.$$

Expanding $1/(1-x^2)$ as a power series and integrating by parts term by term, we get

$$I = \sum_{n=0}^{\infty} \int_0^1 x^{2n}(\log x)dx = -\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}.$$

Thus

$$I = -\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = -\left(1 - \frac{1}{2^2}\right)\zeta(2). \tag{2.3}$$

Changing x to $1/x$ in the integral leads to

$$I = \int_\infty^1 \frac{\log 1/x}{1-(1/x^2)}\left(-\frac{dx}{x^2}\right) = \int_1^\infty \frac{\log x}{1-x^2}dx$$

so that

$$2I = \int_0^\infty \frac{\log x}{1-x^2}dx. \tag{2.4}$$

Inserting (2.2) into (2.4), we deduce that

$$-2I = \int_0^\infty \int_0^\infty \frac{ydxdy}{(1+y^2)(x^2+y^2)}.$$

In the inner integral, changing x to yx leads to

$$-2I = \int_0^\infty \int_0^\infty \frac{dxdy}{(1+x^2)(1+y^2)} = \pi^2/4 \tag{2.5}$$

since

$$\int_0^\infty \frac{dx}{1+x^2} = [\arctan x]_0^\infty = \pi/2.$$

Combining (2.5) with (2.3) gives the desired result.

3. The formula for $\zeta(2k)$

It is possible to show $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ using the following simple recursion which certainly goes back to Euler.

Theorem 1. *For $k \geq 2$,*

$$(k + (1/2))\zeta(2k) = \sum_{i=1}^{k-1} \zeta(2i)\zeta(2k-2i). \tag{3.1}$$

This gives a recursion from which $\zeta(2k) \in \pi^{2k}\mathbb{Q}$ is easily deduced. Our proof of this relies on the following lemma.

Lemma 2. *For any natural number n,*

$$\sum_{m=1}^{\prime\infty} \frac{1}{m^2-n^2} = 3/4n^2,$$

where the dash on the summation means that we exclude $m = n$.

Proof. The sum is

$$\lim_{N\to\infty} \frac{1}{2n} {\sum}'^{N}_{m=1} \left(\frac{1}{m-n} - \frac{1}{m+n} \right)$$

$$= \lim_{N\to\infty} \frac{1}{2n} \left(-\sum_{j=1}^{n-1} \frac{1}{j} + \sum_{j=1}^{N-n} \frac{1}{j} - \sum_{j=n+1}^{N+n} \frac{1}{j} + \frac{1}{2n} \right).$$

Using the elementary fact that as $N \to \infty$,

$$\sum_{j=1}^{N} \frac{1}{j} = \log N + \gamma + o(1),$$

where γ is Euler's constant, the sum in the brackets is

$$\log \frac{N-n}{N+n} + \frac{3}{2n} + o(1)$$

so that our limit is $3/4n^2$, as claimed. □

We can now prove Theorem 1. Inserting the series for the zeta function, the right hand side of (3.1) is

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \sum_{i=1}^{k-1} \frac{1}{n^{2i}} \frac{1}{m^{2k-2i}}.$$

When $n \neq m$, the innermost sum being a geometric sum, is easily seen to be

$$(n^{2-2k} - m^{2-2k})/(m^2 - n^2).$$

Separating the contribution from $m = n$, we deduce that our sum is

$$(k-1)\zeta(2k) + \sum_{n=1}^{\infty} {\sum}'^{\infty}_{m=1} \left(\frac{1}{n^{2k-2}} - \frac{1}{m^{2k-2}} \right) \frac{1}{m^2 - n^2}.$$

Using the lemma, we see that the double sum contributes $(3/2)\zeta(2k)$, which gives the result as claimed.

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