Student Number

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Queen's University
Department of Mathematics and Statistics

MATH 221

Final Examination: December 13, 2012

Instructor: B. Levit

- **PLEASE NOTE:** Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.
- Simple calculators (type Casio fx-991, gold or blue sticker) are permitted.
- Solve Problems 1–6 listed on p. 2. Each Problem is worth 20 points.
- SHOW YOUR WORK CLEARLY. Correct answers without clear work showing how you got there will not receive full marks.
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Marks: Please do not write in the space below

Problem 1 [ ]	Problem 4 [ ]
Problem 2 [ ]	Problem 5 [ ]
Problem 3 [ ]	Problem 6 [ ]

Total: [120]

Green's Theorem

1. **a)** Let C be a simple closed plane curve bounding a region D in the xy-plane. Suppose that C is oriented so that D appears on the left as one goes along C. Show that the line integral of $\mathbf{F} = x\mathbf{j}$ around C is equal to the area of D.

   b) Let C be the plane curve parametrized by $x(t) = 1 - t^2$, $y(t) = t - t^3$, $-1 \le t \le 1$. Sketch the region D bounded by the curve C. When t increases from -1 to 1, is C oriented as required in Part **a)**?

   c) Find the area of the region D in **Part b)**.

   d) Using Green's theorem, calculate $\int_C (\sin x + 6y)\,dx + (8x + e^y)\,dy$, where C is the curve from **Part b)**.

Divergence Theorem

2. Let W be the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$ and inside the positive octant $x \ge 0$, $y \ge 0, z \ge 0$. The boundary of W is a closed surface S. Suppose that S is oriented outward. Find the flux of the vector field $\mathbf{F} = xy^2\,\mathbf{i} + yz^2\,\mathbf{j} + zx^2\,\mathbf{k}$ out of S.

3. Let W be the region bounded by a closed cylinder S of radius a and length $2L$, with the central axis in the z-axis between $z = -L$ and $z = L$. Suppose that S is oriented outward.

   a) Let $\mathbf{F}(x, y, z) = \log y\,\mathbf{i} + y\,\mathbf{j} + (-z - 1)\mathbf{k}$. Show that $\int_S \mathbf{F} \cdot d\mathbf{A} = 0$.

   b) Find the flux of $\mathbf{F}$ through the top and bottom of the cylinder S.

   c) Use **Parts a)** and **b)** to find the flux of $\mathbf{F}$ through the side of the cylinder S.

4. Denote $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, and let $\mathbf{c} = c_1\mathbf{i} + c_2\mathbf{j} + c_3\mathbf{k}$ be a constant vector field.

   a) Find $\operatorname{div}(\mathbf{r} \times \mathbf{c})$.

   b) Find the flux integral $\int_S (\mathbf{r} \times \mathbf{c}) \cdot d\mathbf{A}$ where S is a sphere or radius R centered at the origin.

Stokes' Theorem

5. **a)** Let C be a circle of radius 1 in the plane $x - y + z = 3$, centered at $(1, -1, 1)$ and oriented clockwise when viewed from the origin. Denote by S the disk enclosed by the circle C. Sketch the circle C.

   b) Find the unit orientation vector $\mathbf{n}$ to the disk S compatible with the given orientation of C.

   c) Compute the line integral

$$\int_C (x^3 + y^3 + z^3)dx + 3xy^2\,dy + 3(y + xz^2)\,dz.$$

6. Let S be the part of the sphere $x^2 + y^2 + z^2 = 4$ contained in the octant $x \le 0$, $y \le 0$, $z \ge 0$, oriented outward, and let C be the boundary of S with the induced orientation (thus, C is a simple closed curve consisting of three arcs). Compute the line integral

$$\int_C y\,dx - x\,dy + z\,dz.$$