Math 221 Queen's University, Department of Mathematics

Vector Calculus, Tutorial 1-Solutions

September 2013

1. Consider the function $f(x, y) = \sin(x - y)$ defined on the plane $\mathbf{R}^2$. Describe the level sets of this function using equations. Describe the graph of this function with equations and words or diagrams.

Evaluate the double integral

$$\int\int_{\mathbf{R}} \sin(x-y)dA, \quad \mathbf{R} = \left\{(x,y)|0 \le x \le \frac{\pi}{2}, \quad 0 \le y \le \frac{\pi}{2}\right\}$$

The function $\sin(x - y)$ is constant along the curve $x - y + c = 0$ for different values of the constant c. Rearrranging this equation gives the equation of a straight line with slope 1 in the x-y plane $y = x + c$. The family of level curves of $\sin(x - y)$ is the family of straight lines with slope 1 and y-intercept c. To visualize the graph of $\sin(x - y)$ we can start at the origin (0,0) and move along the direction which is perpendicular to the family of level curves. This would be along the line $y = -x$. The function $\sin(x-y)$ along this curve starts at 0 and looks like $\sin(-2x) = \sin(2x)$, which is periodic of period π.

$$\begin{aligned}
\int\int_{\mathbf{R}} \sin(x-y)dA &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x-y)dydx \\
&= \int_0^{\frac{\pi}{2}}\int_x^{x-\frac{\pi}{2}} -sin(u)dudx \quad (u = x-y, -du = dy) \\
&= \int_0^{\frac{\pi}{2}} \cos(u)\Big|_x^{x-\frac{\pi}{2}} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{\frac{\pi}{2}} \cos(x - \frac{\pi}{2}) - \cos(x)dx \quad \left(u = x - \frac{\pi}{2}, du = dx\right) \\
&= \left(\sin\left(x - \frac{\pi}{2}\right) - \sin(x)\right)\Big|_0^{\frac{\pi}{2}} \\
&= \sin(0) - \sin\left(-\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) + \sin(0) \\
&= +1 - 1 = 0
\end{aligned}
$$

If we integrate in the other order

$$
\begin{aligned}
\int\int_{\mathbf{R}} \sin(x-y)dA &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(x-y)dxdy \\
&= \int_0^{\frac{\pi}{2}}\int_{-y}^{\frac{\pi}{2}-y} sin(u)dudx \quad (u = x - y, du = dx) \\
&= \int_0^{\frac{\pi}{2}} -\cos(u)\Big|_{-y}^{\frac{\pi}{2}-y} dy \\
&= \int_0^{\frac{\pi}{2}} \cos(-y) - \cos(\frac{\pi}{2} - y)dy \quad \left(u = \frac{\pi}{2} - y, -du = dy\right) \\
&= \left(\sin(y) + \sin\left(\frac{\pi}{2} - y\right)\right)\Big|_0^{\frac{\pi}{2}} \\
&= \sin\left(\frac{\pi}{2}\right) - \sin(0) + \sin(0) - \sin\left(\frac{\pi}{2}\right) \\
&= +1 - 1 = 0
\end{aligned}
$$

2. Find the volume of the solid that lies under the plane $4x + 6y - 2z + 15 = 0$ and above the rectangle

$$
\mathbf{R} = \{(x, y)| - 1 \leq x \leq 2, \quad -1 \leq y \leq 1\}
$$

For this problem we only need find the function whose graph corresponds with the plane $4x + 6y - 2z + 15 = 0$. This is clearly

$$
z = f(x, y) = \frac{1}{2}(4x + 6y + 15)
$$

We then compute the signed volume by integrating the function $f(x, y)$ over the domain **R**.

$$\begin{aligned}
\text{signed volume} &= \int\int_{\mathbf{R}} \frac{1}{2}(4x + 6y + 15)\, dA \\
&= \int_{-1}^{2}\int_{-1}^{1} \frac{1}{2}(4x + 6y + 15)\, dydx \\
&= \int_{-1}^{2} \left(2xy + \frac{3}{2}y^2 + 15y\right)\Big|_{-1}^{1}\, dx \\
&= \int_{-1}^{2} \left(4x + \frac{3}{2}0 + 30\right) dx \\
&= \left(2x^2 + 30x\right)\Big|_{-1}^{2} \\
&= 2(4-1) + 30(2+1) \\
&= 96
\end{aligned}$$

3. Evaluate the double integral $\int\int_{\mathbf{D}} x\cos(y)dA$ where **D** is bounded by the lines $y = 0, x = 1$ and the curve $y = x^2$.

We give both solutions to this problem. Notice that the curve $y = x^2$ is equivalent to the curve $x = \sqrt{y}$ in the first quadrant, when both x,y are nonnegative. The region **D** in question lies to the right of the parabola $x = \sqrt{y}$ in the first quadrant, and bounded on the right by the vertical line x=1.

Solution 1 Integrating first with respect to x, keeping y fixed and then with respect to y we find

$$\int\int_{\mathbf{D}} x\cos(y)dA = \int_0^1\int_{\sqrt{y}}^1 x\cos(y)dxdy$$

$$
\begin{aligned}
&= \int_0^1 \frac{1}{2}x^2 \Big|_{\sqrt{y}}^1 \cos(y)dy \\
&= \frac{1}{2}\int_0^1 (1-y)\cos(y)dy \quad (u = (1-y), dv = \cos(y)dy) \\
&= \frac{1}{2}\left((1-y)\sin(y)\Big|_0^1 + \int_0^1 \sin(y)dy\right) \\
&= -\frac{1}{2}(\cos(1) - 1)
\end{aligned}
$$

where we used integration by parts in the fourth line.

Solution2 Interchanging the order of integration

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\begin{aligned}
\int\int_{\mathbf{D}} x\cos(y)dA &= \int_0^1 \int_0^{x^2} x\cos(y)dydx \\
&= \int_0^1 +x\sin(x^2)dx \\
&= \frac{1}{2}\int_0^1 \sin(u) \\
&= -\frac{1}{2}\cos(u)\Big|_0^1 \\
&= -\frac{1}{2}(\cos(1) - 1)
\end{aligned}
$$