Math 221 Queen's University, Department of Mathematics

Vector Calculus, tutorial 5

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1. For the function $f(x,y) = 2x^2 + y^2$

a) Compute the family of flow lines for the gradient field $\vec{F}(x,y) = -\nabla f(x,y)$.

We begin by computing the negative gradient vector field for the potential function $f(x,y) = 2x^2 + y^2$.

$$-\nabla f(x,y) = -4x\vec{i} - 2y\vec{j}$$

For the lines of steepest descent we need to consider the differential equation $\frac{d\vec{r}(t)}{dt} = -\nabla f(\vec{r}(t)$ which in components amounts to the scalar differential equations and solutions (for arbitrary constants a,b)

$$\frac{dx}{dt} = -4x, \quad x(t) = ae^{-4t}, \quad \frac{dy}{dt} = -2y, \quad y(t) = be^{-2t}$$

b) Sketch some of the level curves of the function $f(x,y)$ together with some of the lines of steepest descent for this function.

To sketch the level curves and lines of steepest descent, we can first sketch the family of elliptic curves $2x^2 + y^2 = c^2$ which are ellipses which have semimajor axis along the y-axis and minor axis along the x-axis. The lines of steepest descent are parabolas, which open out along the x-axis. This can be seen by eliminating the time parameter t, between the flow lines. This amounts to the equations

$$\frac{x}{a} = \left(\frac{y}{b}\right)^2$$

which are indeed parabolas as descibed above. To sketch these curves we should include the orientation coming from the vector field $-\nabla f$. This orientation is always directed towards the origin in this example, along these curves, since the flow lines correspond to decreasing values of the function $f(x, y)$.

2. For the vector field $\vec{F} = x^2 i + xyj + 2zk$, and the curve which is the intersection of the cylinder $(x-1)^2 + y^2 = 2$ with the plane $x + y + z = 4$, calculate the work done in moving a particle one complete cycle of the curve, with counterclockwise orientation (as viewed from above the plane).

First we parameterize the circle $(x-1)^2+y^2$ in the plane $z = 0$. We use vector addition to conclude that the parameterized circle is $\vec{r_c}(t) = (1 + \sqrt{2}\cos(t), \sqrt{2}\sin(t)), \quad 0 \leq t \leq 2\pi$. Next we can see how the curve of intersection $\vec{r_i}(t)$ projects onto this circle. Namely the part of the plane lying over the circle $\vec{r_c}$ is the parameterized curve

$$\vec{r_i}(t) = \left(1 + \sqrt{2}\cos(t), \sqrt{2}\sin(t), 3 - \sqrt{2}\cos(t) - \sqrt{2}\sin(t)\right), \quad 0 \leq t \leq 2\pi$$

For this curve we have

$$\frac{d\vec{r_c}(t)}{dt} = \left(-\sqrt{2}\sin(t), \sqrt{2}\cos(t), \sqrt{2}\sin(t) - \sqrt{2}\cos(t)\right)$$

Along the curve $\vec{r}_i(t)$ we have

$$x^2(t) = \left(1+\sqrt{2}\cos(t)\right)^2, \;\; xy = \left(1+\sqrt{2}\cos(t)\right)\sqrt{2}\sin(t), \;\; z(t) = 3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)$$

$$\begin{aligned}
\int_C \vec{F}\cdot\vec{T}ds &= \int_0^{2\pi} \vec{F}\cdot\frac{d\vec{r}_i}{dt}dt \\
&= \int_0^{2\pi} \left(1+\sqrt{2}\cos(t)\right)^2\left(-\sqrt{2}\sin(t)\right)dt \\
&+ \int_0^{2\pi} \left(1+\sqrt{2}\cos(t)\right)\left(\sqrt{2}\sin(t)\right)\left(\sqrt{2}\cos(t)\right)dt \\
&+ \int_0^{2\pi} 2\left(3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)\right)\left(\sqrt{2}\sin(t)-\sqrt{2}\cos(t)\right)dt \\
&= 0
\end{aligned}$$

Each of the three integrals in the above calculation for work are independently 0. In the first two integrals, substitute $u = \sqrt{2}\cos(t), du = -\sqrt{2}\sin(t)dt$. For the last of the three integrals, substitute $u = \left(3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)\right), du = \left(\sqrt{2}\sin(t)-\sqrt{2}\cos(t)\right)dt$

3. Consider the vector field $\vec{F} = 3\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + \vec{\mathbf{k}}$.

a) Show that $\vec{F}$ is a gradient field.

Since we havent yet discovered a method to show that the vector field is conservative without actually constructing the potential function, we will do that in this example.

To show that $\vec{F}$ is a gradient field, we need to construct a function f(x,y,z) so that

$$\frac{\partial f}{\partial x} = 3, \;\; \frac{\partial f}{\partial y} = -2, \;\; \frac{\partial f}{\partial z} = 1$$

Each of these equations are easily integrated and when we compare the results we find that $f(x,y,z) = 3x - 2y + z$

b) Describe the equipotential surfaces of $\vec{F}$ in words and with sketches

The equipotential surfaces of this function are the surfaces S_c where $3x-2y+z=c$ for every constant value c. These surfaces are all planar surfaces with normal direction $\nabla\vec{F} = 3\vec{\mathbf{i}} - 2\vec{\mathbf{j}} + \vec{\mathbf{k}}$ They can be described as the family ofl parallel planes with fixed normal direction.

c) Calculate the work done by the vector field $\vec{F}$ in moving a particle along the parameterized path $\vec{r}(t) = (2,-5,7) + t(-2,-4,6), \quad -1 \leq t \leq 1.$

Since the field is conservative we can calculate the work done as a path integral, which equals the difference in the potential function between the endpoints of the given curve $\vec{r}(-1) = (4,-1,1)$ and $\vec{r}(1) = (0,-9,13)$

$\int_C \vec{F}\cdot d\vec{r} = f(0,-9,13) - f(4,-1,1) = (18+13) - (12+2+1) = 16$ joules $= 16$N-m