Math 221 Queen's University, Department of Mathematics

Vector Calculus, tutorial 6-solutions

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1. For the parameterized helix $\mathcal{C}$, given by $\vec{r}(t) = \cos(t)\vec{\mathbf{i}} + \sin(t)\vec{\mathbf{j}} + t\vec{\mathbf{k}}$, on the time interval $0 \le t \le 1.25\pi$, calculate the path integral

$$\int_{\mathcal{C}} yz^2 e^{xyz^2} dx + xz^2 e^{xyz^2} dy + 2xyze^{xyz^2} dz.$$

To calculate the path integral looks very difficult, even with the parameterization of the path $\mathcal{C}$. In addition the components of the field $\vec{F} = yz^2e^{xyz^2}\vec{\mathbf{i}} + xz^2e^{xyz^2}\vec{\mathbf{j}} + 2xyze^{xyz^2}\vec{\mathbf{k}}$, can be shown to match the partial derivatives of the potential function $f(x, y, z) = e^{xyz^2}$.

$$\frac{\partial f}{\partial x} = yz^2e^{xyz^2}, \quad \frac{\partial f}{\partial y} = xz^2e^{xyz^2}, \quad \frac{\partial f}{\partial z} = 2xyze^{xyz^2}$$

This verifies directly that the field $\vec{F}$ is conservative, and therefore the path integral corresponds to the work done by the gradient field between the endpoints of the helical path $\mathcal{C}$

$$\vec{r}(1.25\pi) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{5\pi}{4}\right), \quad \vec{r}(0) = (1, 0, 0)$$

$$\int_{\mathcal{C}} yz^2e^{xyz^2}dx + xz^2e^{xyz^2}dy + 2xyze^{xyz^2}dz = f(\vec{r}(1.25\pi)) - f(\vec{r}(0)) = e^{\frac{1}{2}\left(\frac{5\pi}{4}\right)^2} - e^0(\text{ joules})$$

2. Consider the vector field $\vec{F} : \mathbb{R} \times (0, +\infty) \to \mathbb{R}^2$ given by

$$\vec{F}(x, y) = \frac{x + xy^2}{y^2}\vec{\mathbf{i}} - \frac{x^2 + 1}{y^3}\vec{\mathbf{j}}.$$

a) Determine whether $\vec{F}$ is a gradient field or not, and give an explanation of your conclusion.

We check the condition for exactness of the vector field $\vec{F}$.

$$\frac{\partial}{\partial y}\left(\frac{x + xy^2}{y^2}\right) = -\frac{2x}{y^3} = \frac{\partial}{\partial x}\left(-\frac{x^2 + 1}{y^3}\right)$$

The vector field is exact and the domain of the vector field $\mathrm{dom}\vec{F} = \mathbb{R} \times (0, +\infty)$ is simply connected. Every closed curve C in this domain, encircles a region R which is entirely conatained in $\mathrm{dom}\vec{F}$. By the theorem we have proved in class, we can conclude that the vector field $\vec{F}$ is conservative on the domain $\mathbb{R} \times (0, +\infty)$.

b) Calculate the work done in moving a particle along the curve $y = 1 + x - x^2$ from $(0, 1)$ to $(1, 1)$.

We will first construct the potential function whose gradient coincides with the vector field $\vec{F}$. Then we can calculate the work done by the conservative force $\vec{F}$ by taking the difference of the potential at the endpoints of the oriented curve $y = 1 + x - x^2$ from $(0, 1)$ to $(1, 1)$.

$$f(x, y) = \int \left(\frac{x + xy^2}{y^2}\right) dx$$

$$
\begin{aligned}
&= \frac{1}{2}\frac{x^2}{y^2} + \frac{1}{2}x^2 + h(y) \\
\frac{\partial f}{\partial y} &= -\frac{x^2}{y^3} + h'(y) \\
&= \left(-\frac{x^2+1}{y^3}\right) \\
h'(y) &= -\frac{1}{y^3} \\
h(y) &= \frac{1}{2y^2} \\
f(x,y) &= \frac{1}{2}\frac{x^2}{y^2} + \frac{1}{2}x^2 + \frac{1}{2y^2}
\end{aligned}
$$

It only remains to compute the work using the fundamental theorem

$$\int_C \vec{F}\cdot d\vec{r} = f(1,1) - f(1,0) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = 1(\text{joule})$$

3)Let $\vec{F} = (3x^2y + y^3 + e^x)\,\vec{\mathbf{i}} + \left(e^{y^2} + 12x\right)\vec{\mathbf{j}}$. Consider the line integral of $\vec{F}$ around the circle of radius a, centered at the origin and oriented counterclockwise.

a) Find the line integral for a=1.

The vector field $\vec{F}$ looks complicated enough on the cirlce of radius a, to attempt a calculation using Green's Theorem, rather than a direct calculation of the circulation of the vector field around the boundary of the circle. For this purpose we have

$$
\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_0^{2\pi}\int_0^1 \left(12 - 3r^2\right) r\,dr\,d\theta \\
&= 12\pi - \frac{6\pi}{4} \\
&= \frac{21\pi}{2}
\end{aligned}
$$

b) For which value of a is the line integral a maximum. Give a clear explanation of your conclusion.

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= \int_0^{2\pi}\int_0^a \left(12 - 3r^2\right) r dr d\theta \\
&= 12\pi a^2 - \frac{6\pi}{4}a^4 \\
\frac{d}{da}\int_C \vec{F}\cdot d\vec{r} &= 24\pi a - 6\pi a^3 \\
&= 6\pi a\left(4 - a^2\right)
\end{aligned}$$

The circulation of the vector field around the counterclockwise circe of radius a , reaches a maximum value when a=2.