Math 221 Queen's University, Department of Mathematics

Vector Calculus, tutorial 7-Solutions

November 2013

1. The electric field $\vec{E}$, at the point with position vector $\vec{r}$ in $\mathbb{R}^3$, due to a charge q at the origin is given by

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{\|\vec{r}\|^3},$$

a) Compute curl $\vec{E}$. Is $\vec{E}$ a path independent vector field? Give a clear explanation of your conclusion.

The electric field in components is

$$\vec{E}(\vec{r}) = q\frac{\vec{r}}{\|\vec{r}\|^3} = \left(\frac{qx}{[x^2+y^2+z^2]^{\frac{3}{2}}}, \frac{qy}{[x^2+y^2+z^2]^{\frac{3}{2}}}, \frac{qz}{[x^2+y^2+z^2]^{\frac{3}{2}}}\right)$$

By symmetry we can see (without actually doing the computation)

$$\begin{aligned}
\frac{\partial}{\partial y}\frac{qz}{[x^2+y^2+z^2]^{\frac{3}{2}}} &= \frac{\partial}{\partial z}\frac{qy}{[x^2+y^2+z^2]^{\frac{3}{2}}}\\
\frac{\partial}{\partial x}\frac{qz}{[x^2+y^2+z^2]^{\frac{3}{2}}} &= \frac{\partial}{\partial z}\frac{qx}{[x^2+y^2+z^2]^{\frac{3}{2}}}\\
\frac{\partial}{\partial x}\frac{qy}{[x^2+y^2+z^2]^{\frac{3}{2}}} &= \frac{\partial}{\partial y}\frac{qx}{[x^2+y^2+z^2]^{\frac{3}{2}}}
\end{aligned}$$

From this it immediately follows that the three components of the vector curl $\vec{E}$ are identically zero.

The domain of the electric field $\vec{E}$ is $\mathbb{R}^3/\{(0,0,0)\}$, which means all of $\mathbb{R}^3$ excluding the singular point at (0,0,0). This domain is simply connected in $\mathbb{R}^3$ which means that every simple closed curve can be continuously deformed to a point without leaving the domain of the vector field $\vec{E}$. Thus by the converse to the theorem on the curl test (described in class),we can conclude that there is a potential function, and the electric field $\vec{E}$ is conservative in its domain and thus path independent.

b) If possible, find a potential function for $\vec{E}$.

To construct a potential function, it is neccessary that we find the function $f(x, y, z)$ so that

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{qx}{[x^2 + y^2 + z^2]^{\frac{3}{2}}}\\ \frac{\partial f}{\partial y} &= \frac{qy}{[x^2 + y^2 + z^2]^{\frac{3}{2}}}\\ \frac{\partial f}{\partial z} &= \frac{qz}{[x^2 + y^2 + z^2]^{\frac{3}{2}}}\end{aligned}$$

This function is $f(x, y, z) = \frac{-q}{[x^2+y^2+z^2]^{\frac{1}{2}}}$ which can be easily verified.

2. Calculate the area of the bounded region inside the folium of Descartes, $x^3 + y^3 = 3xy$.

a)Sketch the bounded region and show that this region has a boundary which is parameterized by the vector function $\vec{r}(t) : [0, \infty) \to \mathbb{R}^2$

$$\vec{r}(t) = \frac{3t}{1 + t^3}\vec{\mathbf{i}} + \frac{3t^2}{1 + t^3}\vec{\mathbf{j}}$$

The folium of Descartes is the beautiful closed oval shaped loop (pinched at (0,0)) in the first quadrant of the x=y plane. We can use Green's theorem to conclude that this area enclosed by this loop may be calculated

$$\text{Area } R = \int\int_R dA = \int_{\partial R} -y dx = \int_{\partial R} x dy$$

a)Sketch the bounded region and show that this region has a boundary which is parameterized by the vector function $\vec{r}(t) : [0, \infty) \to \mathbb{R}^2$

$$\vec{r}(t) = \frac{3t}{1+t^3}\vec{\mathbf{i}} + \frac{3t^2}{1+t^3}\vec{\mathbf{j}}$$

To show this we need only calculate the terms x^3, y^3 using the components of the given parameterization $\vec{r}(t)$

$$\begin{aligned} x^3 + y^3 &= \frac{27t^3}{(1+t^3)^3} + \frac{27t^6}{(1+t^3)^3} \\ &= \frac{27(t^3+t^6)}{(1+t^3)^3} \\ &= \frac{27t^3(1+t^3)}{(1+t^3)^3} \\ &= \frac{27t^3}{(1+t^3)^2} \\ &= 3xy \end{aligned}$$

Next notice that when t=0, $\vec{r}(0) = (0,0)$ and as $t \to \infty, \vec{r}(t) \to (0,0)$. Finally we observe that the orientation on the folium of Descartes given by the vector function $\vec{r}(t)$ is counterclockwise, or positive orientation. This follows from the fact that for $0 < t < 1, x > y$ and for $1 < t < \infty, x < y$.

b) Using this parameterization and Green's Theorem calculate the area of the bounded region.

From the comment at the beginning of the question (using Green's theorem)

$$\begin{aligned}
\text{Area } R &= \int_{\partial R} x dy \\
&= \int_0^\infty \left(\frac{3t}{1+t^3}\right)\left(\frac{6t}{1+t^3} - \frac{3t^2\left(3t^2\right)}{\left(1+t^3\right)^2}\right) dt \\
&= \int_0^\infty \left(\frac{9t^2\left(2-t^3\right)}{\left(1+t^3\right)^3}\right) dt
\end{aligned}$$

Setting $u = 1 + t^3$ gives $du = 3t^2 dt$ and

$$\begin{aligned}
\int_0^\infty \left(\frac{9t^2\left(2-t^3\right)}{\left(1+t^3\right)^3}\right) dt &= \int_1^\infty \frac{3(2-(u-1))}{u^3} du \\
&= \int_1^\infty \left(9u^{-3} - 3u^{-2}\right) du \\
&= \lim_{m\to\infty}\left[-\frac{9}{2}u^{-2} + 3u^{-1}\right]_1^m \\
&= \lim_{m\to\infty}\left[\left(-\frac{9}{2m^2} + \frac{3}{m}\right) - \left(-\frac{9}{2} + 3\right)\right] \\
&= \frac{3}{2} \text{ (wow!)}
\end{aligned}$$

The area within the folium of Descartes is 3/2 (bet you didnt see that one coming!).

3) The donut shaped surface **S** (called a torus)

$$\left(\sqrt{x^2+y^2} - a\right)^2 + z^2 = b^2, \quad a > b > 0$$

can be parameterized by $\mathbf{T} : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$.

$$\mathbf{T}(\theta, \phi) = (a + b\cos(\theta))\cos(\phi)\vec{\mathbf{i}} + (a + b\cos(\theta))\sin(\phi)\vec{\mathbf{j}} + b\sin(\theta)\vec{\mathbf{k}}$$

Find the surface area of the torus **S**.

b) Calculate the area of the ellipse **E** on the plane $2x + y + z = 2$ cut out by the circular cylinder $x^2 + y^2 = 2x$.