Solutions #10

1. **(a)** Find a parametrization for the hyperboloid $x^2 + y^2 - z^2 = 25$;
(b) Find an expression for a unit normal to this surface.
(c) Find an equation for the plane tangent to the hyperboloid at the point $(a, b, 0)$ where $a^2 + b^2 = 25$.
(d) Show that the lines $t \mapsto (a - tb, b + ta, 5t)$ and $t \mapsto (a + tb, b - ta, 5t)$ lie in the surface and in the tangent plane found in part (c).

Solution.

(a) In cylindrical coordinates, we have $r^2 - z^2 = 25$. Since $r \geq 0$, it follows that $r = \sqrt{25 + z^2}$. Hence, the parametrization $\vec{\boldsymbol{\sigma}} \colon [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$ with

$$\vec{\boldsymbol{\sigma}}(\theta, z) = \sqrt{25 + z^2}\cos(\theta)\vec{\boldsymbol{\imath}} + \sqrt{25 + z^2}\sin(\theta)\vec{\boldsymbol{\jmath}} + z\vec{\boldsymbol{k}}$$

is one possible solution.

(b) The standard normal vector is

$$\begin{aligned}\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta} \times \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial z} &= \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ -\sqrt{25 + z^2}\sin(\theta) & \sqrt{25 + z^2}\cos(\theta) & 0 \\ \frac{z\cos(\theta)}{\sqrt{25+z^2}} & \frac{z\sin(\theta)}{\sqrt{25+z^2}} & 1 \end{bmatrix} \\ &= \sqrt{25 + z^2}\cos(\theta)\vec{\boldsymbol{\imath}} + \sqrt{25 + z^2}\sin(\theta)\vec{\boldsymbol{\jmath}} - z\vec{\boldsymbol{k}}\,.\end{aligned}$$

Since $\left\| \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta} \times \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial z} \right\| = \sqrt{(25 + z^2) + (-z)^2} = \sqrt{25 + 2z^2}$, the vector

$$\frac{\sqrt{25 + z^2}\cos(\theta)}{\sqrt{25 + 2z^2}}\vec{\boldsymbol{\imath}} + \frac{\sqrt{25 + z^2}\sin(\theta)}{\sqrt{25 + 2z^2}}\vec{\boldsymbol{\jmath}} - \frac{z}{\sqrt{25 + 2z^2}}\vec{\boldsymbol{k}}$$

is a unit normal to the surface.

(c) If $f(x, y, z) = x^2 + y^2 - z^2 - 25$, then the hyperboloid is the level surface at height 0 and the tangent plane at the point $(a, b, 0)$ is

$$\begin{aligned} f_x(a, b, 0)(x - a) + f_y(a, b, 0)(y - b) + f_z(a, b, 0)(z - 0) &= 0 \\ 2a(x - a) + 2b(y - b) - 2(0)(z) &= 0 \\ ax + by = a^2 + b^2 &= 25\,. \end{aligned}$$

(d) Since $a(a \pm tb) + b(b \mp ta) = a^2 \pm tab + b^2 \mp tab = a^2 + b^2 = 25$ and

$$\begin{aligned}(a \pm tb)^2 + (b \mp ta)^2 - (5t)^2 &= a^2 \pm 2tab + t^2b^2 + b^2 \mp 2tab + t^2a^2 - 25t^2 \\ &= (a^2 + b^2) + t^2(a^2 + b^2 - 25) = 25\,,\end{aligned}$$

if follows that the two lines lie on the hyperboloid and the tangent plane. □

Alternative Solution for 1(c). From the parametrization in part (a), we see that the point $(a, b, 0)$ where $a^2 + b^2 = 25$ corresponds to the point $(\theta, z) = (\theta, 0)$. In particular, we have $a = 5\cos(\theta)$ and $b = 5\sin(\theta)$. Hence, Part (b) implies that the normal vector to the tangent plane at the point $(a,b,0)$ is $\sqrt{25+0^2}\cos(\theta)\vec{\boldsymbol{\imath}} + \sqrt{25+0^2}\sin(\theta)\vec{\boldsymbol{\jmath}} - 2(0)\vec{\boldsymbol{k}} = a\vec{\boldsymbol{\imath}} + b\vec{\boldsymbol{\jmath}}$. Therefore, the tangent plane at the point $(a,b,0)$ is

$$(a\vec{\boldsymbol{\imath}} + b\vec{\boldsymbol{\jmath}}) \cdot \big((x-a)\vec{\boldsymbol{\imath}} + (y-b)\vec{\boldsymbol{\jmath}} + (z-0)\vec{\boldsymbol{k}}\big) = 0 \quad \Longrightarrow \quad ax + by = a^2 + b^2 = 25\,. \qquad \square$$

2. Let $\vec{\boldsymbol{H}}(x,y,z) := (e^{xy} + 3z + 5)\vec{\boldsymbol{\imath}} + (e^{xy} + 5z + 3)\vec{\boldsymbol{\jmath}} + (3z + e^{xy})\vec{\boldsymbol{k}}$. Calculate the flux of $\vec{\boldsymbol{H}}$ through the square S of side length 2 with one vertex at the origin, one edge along the positive y-axis, one edge in the xz-plane with $x > 0$, $z > 0$ and normal $\vec{\boldsymbol{n}} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{k}}$.

Solution. Since the unit normal to S is $\frac{1}{\sqrt{2}}\vec{\boldsymbol{n}}$, we have

$$\begin{aligned}\int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} &= \int_S \vec{\boldsymbol{H}} \cdot \frac{1}{\sqrt{2}}\vec{\boldsymbol{n}}\, dA = \frac{1}{\sqrt{2}}\int_S (e^{xy} + 3z + 5) - (3z + e^{xy})\, dA \\ &= \frac{5}{\sqrt{2}}\int_S dA = \frac{5}{\sqrt{2}}\mathrm{Area}(S) = \frac{5}{\sqrt{2}}(2^2) = 10\sqrt{2}\,. \qquad \square\end{aligned}$$

Alternative Solution. Since the distance between the origin and the point $(\sqrt{2}, 0, \sqrt{2})$ is two units, S is parametrize by $\vec{\boldsymbol{\sigma}} \colon [0,2] \times [0,\sqrt{2}] \to \mathbb{R}^3$ with $\vec{\boldsymbol{\sigma}}(u,v) := v\vec{\boldsymbol{\imath}} + u\vec{\boldsymbol{\jmath}} + v\vec{\boldsymbol{k}}$. The standard normal vector is

$$\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial u} \times \frac{\partial\vec{\boldsymbol{\sigma}}}{\partial v} = \det\begin{bmatrix}\vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ 0 & 1 & 0 \\ 1 & 0 & 1\end{bmatrix} = \vec{\boldsymbol{\imath}} - \vec{\boldsymbol{k}}\,.$$

Thus, we have

$$\begin{aligned}\int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} &= \int_0^{\sqrt{2}}\int_0^2 \vec{\boldsymbol{H}}\big(\vec{\boldsymbol{\sigma}}(u,v)\big) \cdot \left(\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial u} \times \frac{\partial\vec{\boldsymbol{\sigma}}}{\partial v}\right) du\, dv \\ &= \int_0^{\sqrt{2}}\int_0^2 \big((e^{uv} + 3v + 5)\vec{\boldsymbol{\imath}} + (e^{uv} + 5v + 3)\vec{\boldsymbol{\jmath}} + (3v + e^{uv})\vec{\boldsymbol{k}}\big) \cdot \big(\vec{\boldsymbol{\imath}} - \vec{\boldsymbol{k}}\big)\, du\, dv \\ &= \int_0^{\sqrt{2}}\int_0^2 5\, du\, dv = 5(2)(\sqrt{2}) = 10\sqrt{2}\,. \qquad \square\end{aligned}$$

3(a). The torus T can be parametrized by $\vec{\boldsymbol{r}} \colon [0,2\pi) \times [0,2\pi) \to \mathbb{R}^3$ where $a > b > 0$ and

$$\vec{\boldsymbol{r}}(\theta,\phi) = \big(a + b\cos(\theta)\big)\cos(\phi)\vec{\boldsymbol{\imath}} + \big(a + b\cos(\theta)\big)\sin(\phi)\vec{\boldsymbol{\jmath}} + b\sin(\theta)\vec{\boldsymbol{k}}\,.$$

Find the surface area of T.

Solution. The standard normal vector to T is

$$\begin{aligned}
\frac{\partial \vec{\boldsymbol{\tau}}}{\partial \theta} \times \frac{\partial \vec{\boldsymbol{\tau}}}{\partial \phi} &= \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ -b\sin(\theta)\cos(\phi) & -b\sin(\theta)\sin(\phi) & b\cos(\theta) \\ -\big(a + b\cos(\theta)\big)\sin(\phi) & \big(a + b\cos(\theta)\big)\cos(\phi) & 0 \end{bmatrix} \\
&= -b\big(a + b\cos(\theta)\big) \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ \sin(\theta)\cos(\phi) & \sin(\theta)\sin(\phi) & -\cos(\theta) \\ -\sin(\phi) & \cos(\phi) & 0 \end{bmatrix} \\
&= -b\big(a + b\cos(\theta)\big)\big(\cos(\theta)\cos(\phi)\vec{\boldsymbol{\imath}} + \cos(\theta)\sin(\phi)\vec{\boldsymbol{\jmath}} + \sin(\theta)\vec{\boldsymbol{k}}\big)\,,
\end{aligned}$$

so its magnitude is

$$\left\| \frac{\partial \vec{\boldsymbol{\tau}}}{\partial \theta} \times \frac{\partial \vec{\boldsymbol{\tau}}}{\partial \phi} \right\| = b\big(a + b\cos(\theta)\big)\big\|\cos(\theta)\cos(\phi)\vec{\boldsymbol{\imath}} + \cos(\theta)\sin(\phi)\vec{\boldsymbol{\jmath}} + \sin(\theta)\vec{\boldsymbol{k}}\big\| = b\big(a + b\cos(\theta)\big)\,.$$

Consequently, the surface area of the torus is

$$\begin{aligned}
\operatorname{Area}(T) &= \int_T 1 \; dA = \int_0^{2\pi}\!\!\int_0^{2\pi} \left\| \frac{\partial \vec{\boldsymbol{\tau}}}{\partial \theta} \times \frac{\partial \vec{\boldsymbol{\tau}}}{\partial \phi} \right\| \; d\theta\, d\phi \\
&= \int_0^{2\pi}\!\!\int_0^{2\pi} b\big(a + b\cos(\theta)\big) \; d\theta\, d\phi = b\left(\int_0^{2\pi} \; d\phi\right)\left(\int_0^{2\pi} a + b\cos(\theta) \; d\theta\right) \\
&= 2\pi b\big[a\theta + b\sin(\theta)\big]_0^{2\pi} = 4\pi^2 ab\,.
\end{aligned}$$

□

3(b). Find the area of the ellipse E on the plane $2x + y + z = 2$ cut out by the circular cylinder $x^2 + y^2 = 2x$.

Solution. By completing the square $x^2 - 2x + y^2 = (x-1)^2 - 1 + y^2$, we see that the cylinder is given by $(x-1)^2 + y^2 = 1$. Hence, the region bounded by the cylinder is parameterized $\vec{\boldsymbol{\xi}}\colon [0,1] \times [0, 2\pi] \times (-\infty, \infty) \to \mathbb{R}^3$ where

$$\vec{\boldsymbol{\xi}}(r, \theta, w) := \big(r\cos(\theta) + 1\big)\vec{\boldsymbol{\imath}} + r\sin(\theta)\vec{\boldsymbol{\jmath}} + w\vec{\boldsymbol{k}}\,.$$

It follows that the intersection E of this region with the plane $z = 2 - 2x - y$ is parameterized by $\vec{\boldsymbol{\sigma}}\colon [0,1] \times [0, 2\pi] \to \mathbb{R}^3$ where

$$\vec{\boldsymbol{\sigma}}(r, \theta) := \big(r\cos(\theta) + 1\big)\vec{\boldsymbol{\imath}} + r\sin(\theta)\vec{\boldsymbol{\jmath}} + \big(-2r\cos(\theta) - r\sin(\theta)\big)\vec{\boldsymbol{k}}\,.$$

Since

$$\begin{aligned}
\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial r} \times \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta} &= \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ \cos(\theta) & \sin(\theta) & -2\cos(\theta) - \sin(\theta) \\ -r\sin(\theta) & r\cos(\theta) & 2r\sin(\theta) - r\cos(\theta) \end{bmatrix} \\
&= \big(2r\sin^2(\theta) - r\sin(\theta)\cos(\theta) + 2r\cos^2(\theta) + r\sin(\theta)\cos(\theta)\big)\vec{\boldsymbol{\imath}} \\
&\quad - \big(2r\sin(\theta)\cos(\theta) - r\cos^2(\theta) - 2r\sin(\theta)\cos(\theta) - r\sin^2(\theta)\big)\vec{\boldsymbol{\jmath}} \\
&\quad + \big(r\cos^2(\theta) + r\sin^2(\theta)\big)\vec{\boldsymbol{k}} \\
&= 2r\vec{\boldsymbol{\imath}} + r\vec{\boldsymbol{\jmath}} + r\vec{\boldsymbol{k}}
\end{aligned}$$

we have $\left\| \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial r} \times \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta} \right\| = \sqrt{4r^2 + r^2 + r^2} = \sqrt{6}r$ and

$$\text{Area}(E) = \int_E dS = \int_0^1 \int_0^{2\pi} \sqrt{6}r \; d\theta \; dr = \sqrt{6}(2\pi) \left[\frac{1}{2} r^2 \right]_0^1 = \sqrt{6}\pi \,. \qquad \square$$

Alternative Solution. The ellipse E is the graph of the function $f(x, y) = 2 - 2x - y$ over the unit disk D in the xy-plane centered at the point $(1, 0)$. Hence, $\vec{\boldsymbol{\sigma}}(x, y) = x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + f(x, y)\vec{\boldsymbol{k}}$ where $(x, y) \in D$ parametrizes the surface, and we have

$$\frac{\partial \vec{\boldsymbol{\sigma}}}{\partial x} \times \frac{\partial \vec{\boldsymbol{\sigma}}}{\partial y} = \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{bmatrix} = -f_x \vec{\boldsymbol{\imath}} - f_y \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}} = 2\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}} \,.$$

If follows that

$$\int_E dA = \int_D \|2\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\jmath}} + \vec{\boldsymbol{k}}\| \; dA = \sqrt{6} \int_D dA = \sqrt{6}\,\text{Area}(D) = \sqrt{6}\pi(1)^2 = \sqrt{6}\pi \,. \qquad \square$$