Name and Student Number

Math 221, Vector Calculus, Fall 2013-Solutions
Queen's University, Department of Mathematics
Please write your student number and your name clearly at the top of this page.
Do all five questions, each is marked out of 10.

1. a)[**4marks**] Find a parameterization of the positively oriented closed curve (looking down the z-axis) which is the intersection of the plane $x+y+2z=4$ and the vertical cylinder $x^2+y^2=4$.

We need to parameterize this curve C. We can see that the x,y components of this curve are just the circular path around the cylinder $x^2+y^2=4$, oriented counterclockwise, while the z-component can be calculated from the equation of the plane: $z=2-\frac{x}{2}-\frac{y}{2}$. The parameterization of C can be done using polar coordinates

$$x(t)=2\cos(t),\quad y(t)=2\sin(t),\quad z(t)=2-\cos(t)-\sin(t),\quad 0\le t\le 2\pi$$

b)[**6 marks**] Find the circulation of the horizonal vector field $\vec{F}(x,y,z)=(-y,x,0)$ around the closed curve you described in part a)

We calculate the vector field $\vec{F}$ on the parameterized curve $\vec{r}(t)$, and the velocity vector

$$\vec{F}(\vec{r}(t))=(-2\sin(t),2\cos(t),0),\quad \frac{d\vec{r}}{dt}=(-2\sin(t),2\cos(t),\sin(t)-\cos(t))$$

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_0^{2\pi}(-2\sin(t),2\cos(t),0)\cdot(-2\sin(t),2\cos(t),\sin(t)-\cos(t))\,dt\\ &= \int_0^{2\pi}\Big(4\cos^2(t)+4\sin^2(t)\Big)\,dt=8\pi\end{aligned}$$

2 a)[6 marks] For the vector field $\vec{F}(x,y) = (x^2 - y^2, 2xy)$, calculate the work done in moving a particle around the closed boundary of the square of side length 3, with vertices (0,0),(3,0), (3,3), (0,3). Use counterclockwise orientation for this path.

We parameterize the four line segments C_1, C_2, C_3, C_4 which together form the PWS boundary C of the square.

$$\vec{r}_1(t) = (t,0), \vec{r}_2(t) = (3,t), \vec{r}_3(t) = (3-t,3), \vec{r}_4(t) = (0,3-t), \quad 0 \le t \le 3$$

$$\begin{aligned}
\int_{C_1} \vec{F} \cdot d\vec{r}_1 &= \int_0^3 \vec{F}(\vec{r}_1(t)) \cdot \frac{d\vec{r}_1}{dt} dt = \int_0^3 t^2 dt = 9 \\
\int_{C_2} \vec{F} \cdot d\vec{r}_2 &= \int_0^3 \vec{F}(\vec{r}_2(t)) \cdot \frac{d\vec{r}_2}{dt} dt = \int_0^3 6t dt = 27 \\
\int_{C_3} \vec{F} \cdot d\vec{r}_3 &= \int_0^3 \vec{F}(\vec{r}_3(t)) \cdot \frac{d\vec{r}_3}{dt} dt = \int_0^3 -(x^2 - 9) dt = 18 \\
\int_{C_4} \vec{F} \cdot d\vec{r}_4 &= \int_0^3 \vec{F}(\vec{r}_4(t)) \cdot \frac{d\vec{r}_4}{dt} dt = \int_0^3 0 dt = 0
\end{aligned}$$

$$\int_C \vec{F} \cdot d\vec{r} = \sum_{i=1,4} \int_{C_i} \vec{F} \cdot d\vec{r}_i = 54$$

You can also use Green's Theorem to do this example (but I expected the first method)

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \\
&= \int_0^3 \int_0^3 (2y - (-2y)) dy dx \\
&= \int_0^3 \int_0^3 4y dy dx \\
&= \int_0^3 2(9) dx = 54
\end{aligned}$$

b)[4 marks] Is the vector field $\vec{F}$ conservative? Give a clear explanation as to how you reach your conclusion.

This vector field is not conservative. First, the work done around the closed path C is not zero, and therefore the path integrals for $\vec{F}$ are not path independent. We have seen that this implies that $\vec{F}$ is not conservative.
Next we could check the condition for exactness of this field with the components $P = (x^2 - y^2), Q = 2xy$. However we find that

$$\frac{\partial P}{\partial y} = -2y \neq \frac{\partial Q}{\partial x} = 2y$$

Since exactness is not met, the vector field $\vec{F}$ is not conservative.

3. Consider the vector field

$$\vec{F} = \left(3x^2 - y + ye^{xy}\right)\vec{\mathbf{i}} + \left(3y^2 - x + xe^{xy}\right)\vec{\mathbf{j}}$$

a)[**5 marks**] Give a clear explanation as to why or why not this vector field is a gradient field.

First we calculate the partial derivatives used to consider the exactness condition $\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) = 0$.

$$\frac{\partial Q}{\partial x} = -1 + e^{xy} + xye^{xy}, \quad \frac{\partial P}{\partial y} = -1 + e^{xy} + xye^{xy}$$

Since these two are equal, we next need to consider the domain of the vector field $\vec{F}$. Considering each component, we find that $\text{dom}\vec{F} = \mathbf{R}^2$ which is simply connected, and thus the conditions are met to guarantee that the vector field is conservative and path independent. An alternative method is to actually construct the potential function $f(x, y)$ from the partial derivatives

$$\frac{\partial f}{\partial x} = \left(3x^2 - y + ye^{xy}\right), \quad \frac{\partial f}{\partial y} = \left(3y^2 - x + xe^{xy}\right)$$

b)[**5marks**] Calculate the work done by the force $\vec{F}$ in moving a particle along the curve $y = 1 + x - x^2$ between the points (0,1) and (1,1).

For this problem we can do either of the following. First, use path independence of the vector field $\vec{F}$ and construct the function $f(x, y)$ from its partial derivatives. Or we can do the path integral directly, using a parameterization of the curve joining the endpoints (0,1) to (1,1). The second method is much more complicated in this case, due to the exponential terms in the components of $\vec{F}$.

$$\begin{aligned}
f(x, y) &= \int_x \left(3x^2 - y + ye^{xy}\right) dx \\
&= \left(x^3 - xy + \frac{ye^{xy}}{y} + h(y)\right) \\
\frac{\partial f}{\partial y} &= -x + xe^{xy} + h'(y) \\
h'(y) &= \left(3y^2 - x + xe^{xy}\right) - (-x + xe^{xy}) \\
&= 3y^2 \\
h(y) &= y^3 \\
f(x, y) &= x^3 + y^3 - xy + e^{x,y}
\end{aligned}$$

Now we can evaluate the path integral

$$\begin{aligned}
\text{work} &= \int_C \vec{F} \cdot d\vec{r} \\
&= f(1, 1) - f(0, 1) \\
&= (1 + 1 - 1 + e) - (1 + 1) = e - 1 \text{ joules}
\end{aligned}$$

4.[10 marks] Consider the vector field $\vec{F} = (x^2 - 2xy)\,\vec{\mathbf{i}} + (y^2 - 2xy)\,\vec{\mathbf{j}}$. Using Green's Theorem, calculate the path integral $\int_C \vec{F} \cdot d\vec{r}$ around the PWS closed curve C which consists of two segments C_1, C_2 with the indicated orientation. C_1 is a segment of the parabola $y = x^2$, and C_2 is a straight line segment.

We see that the region R between the two curves qualifies for an application of Green's theorem, and the boundary is oriented postively with respect to this region

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \\
&= \int_{-2}^{1}\int_{x^2}^{2-x} (-2y + 2x)dydx \\
&= \int_{-2}^{1} \left(-y^2 + 2xy\right)\Big|_{x^2}^{2-x}\, dx \\
&= \int_{-2}^{1} \left(-4 + 8x - 3x^2 - 2x^3 + x^4\right) dx \\
&= \left(-4x + 4x^2 - x^3 - \frac{1}{2}x^4 + \frac{1}{5}x^5\right)\Big|_{-2}^{1} \\
&= -18.9
\end{aligned}$$

You can also do this problem by parameterizing both curves C_1, C_2. In this case one such parameterization is

$$\vec{r}_1(t) = \left(t, t^2\right), \frac{d\vec{r}_1}{dt} = (1, 2t), -2 \le t \le 1, \quad \vec{r}_2(t) = (1 - 3t, 1 + 3t), \frac{d\vec{r}_2}{dt} = (-3, 3), 0 \le t \le 1.$$

We then calculate the path integrals directly,

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_{-2}^{1} \left(t^2 - 2t^3, t^4 - 2t^3\right) \cdot (1, 2t)dt \\
&+ \int_0^1 \left((1-3t)^2 - 2(1-3t)(1+3t), (1+3t)^2 - 2(1-3t)(1+3t)\right) \cdot (-3, 3)dt
\end{aligned}$$

The integrals are now evaluated in straightforward way, but clearly using Green's theorem is preferable and more easily calculated.

5a) [6 marks] Consider the vector field $\vec{F} = (cxy, x^6y^2)$ for postive parameter $c > 0$. Evaluate the path integral $\int_C \vec{F} \cdot d\vec{r}$ over the parameterized curve $y = ax^b$ for positive parameters a,b, between x=0, and x=1.

The curve $y = ax^b$ can be parameterized with parameter x, $\vec{r}(x) = (x, ax^b), 0 \leq x \leq 1$.

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \int_0^1 \left(cax^{b+1}, x^6a^2x^{2b}\right) \cdot \left(1, abx^{b-1}\right) dx \\
&= \int_0^1 \left(acx^{b+1} + a^3bx^{3b+5}\right) dx \\
&= \frac{ac}{b+2} x^{b+2} \Big|_0^1 + \frac{a^3b}{3b+6} x^{3b+6} \Big|_0^1 \\
&= \frac{ac}{b+2} + \frac{a^3b}{3b+6} \\
&= \frac{3ac + a^3b}{3b+2}
\end{aligned}$$

It should be mentioned here that Green's Theorem is not applicable for this calculation, since the curve C is NOT closed, but instead has separated endpoints (0,0) and (1,a) (independent of the parambeter b).

b)[4marks] Find the value of the constant c, so that this integral is independent of the parameter b. To make the value of the path integral independent of b, we might consider the possibility

first of finding the value of c, which makes the vector field path independent. However it can easily be seen that for any choice of the constant c

$$\frac{\partial Q}{\partial x} = 6x^5y^2 \neq \frac{\partial P}{\partial y} = cx$$

Instead we try to isolate the constant b in the expression for the path integral, and find that if $c = \frac{2a^2}{3}$ then

$$\int_C \vec{F} \cdot d\vec{r} = \frac{2a^3 + a^b}{3b+2} = \frac{a^3}{3}$$

which is independent of the parameter b.