Name and Student Number

Math 221, Vector Calculus, Fall 2013
Queen's University, Department of Mathematics
Solutions to Midterm Exam 1.
Do all five questions, each is marked out of 10.

1. The table gives values of the function $f(x, y)$. Use this table to find upper and lower Riemann sums for $\int\int_{\mathbf{R}} f dA$, over the rectangle

$$\mathbf{R} = \{(x, y) \,|\, 2 \le x \le 3, \quad -3 \le y \le -1\}$$

	x	2.0	2.5	3.0	
y					
-1.0		1	-1	-2	
-2.0		0	3	2	
-3.0		1	7	4	

The table can be converted into a diagram showing that there are four subrectangles in the rectangle $\mathbf{R}$ which can be denoted $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$. Each of these subrectangles has equal area $\Delta A_{i,j} = 0.5$. On each subrectangle we compute the values **M**= maximum f (for the upper sum) , and **m**=minimum f (for the lower sum). These values are tabulated in the following

	m	**M**
$A_{1,1}$	0	7
$A_{1,2}$	2	7
$A_{2,1}$	-1	3
$A_{2,2}$	-2	3

Now we compute the lower Riemann sum

$$\sum\sum \mathbf{m}_{i,j}\Delta A_{i,j} = (0 + 2 - 1 - 2)0.5 = -0.5$$

and the upper Riemann sum

$$\sum\sum \mathbf{M}_{i,j}\Delta A_{i,j} = (7 + 7 + 3 + 3)0.5 = 10$$

2. Find the volume of the region under the graph of the function $f(x, y) = xy$ and above the region in the x-y plane bounded by the lines

$$y = 0, \quad y = x, \quad x + y = 2.$$

$$\begin{aligned}
\text{Volume} &= \int\int_R f(x,y)dA \\
&= \int_0^1 \int_y^{2-y} xydxdy \\
&= \int_0^1 \frac{1}{2}x^2y \,\Big|_y^{2-y}\, dy \\
&= \int_0^1 \frac{y}{2}\left[(2-y)^2 - y^2\right] dy \\
&= \int_0^1 \frac{y}{2}\left[4 - 4y\right] dy \\
&= \int_0^1 [2y - 2y^2]dy \\
&= \left[y^2 - \frac{2}{3}y^3\right] \Big|_0^1 \\
&= \frac{1}{3}
\end{aligned}$$

3. Using triple integrals, find the volume of the region bounded between the planes $x+y+2z=4$ and the x-y plane, and lying above the circular region $x^2+y^2 \leq 4$, in the x-y plane.

$$\begin{aligned}
\text{Volume} &= \int\int\int_W dV \quad (\text{ in cylindrical coodinates}) \\
&= \int_0^{2\pi}\int_0^2\int_0^{2-\frac{r}{2}\cos(\theta)-\frac{r}{2}\sin(\theta)} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2 r\left(2-\frac{r}{2}\cos(\theta)-\frac{r}{2}\sin(\theta)\right) dr\,d\theta \\
&= \int_0^{2\pi}\int_0^2 2r\,dr\,d\theta - \int_0^2\int_0^{2\pi}\frac{r^2}{2}(\cos(\theta)+\sin(\theta))\,d\theta\,dr \\
&= 2\pi r^2\Big|_0^2 - 0 \quad \left(\int_0^{2\pi}\cos(\theta)d\theta = \int_0^{2\pi}\sin(\theta)d\theta = 0\right) \\
&= 8\pi
\end{aligned}$$

4. Find the value of the triple integral $\int\int\int_{\mathbf{W}} x^2 dV$ where $\mathbf{W}$ is the region bounded by the planes

$$z = 4y, \quad z = 2y, \quad y = 1, \quad x = -1, \quad x = 2.$$

The region bounded by the planes can be expressed as

$$-1 \leq x \leq 2, \quad 2y \leq z \leq 4y, \quad 0 \leq y \leq 1$$

We can integrate over this region in the order dxdzdy, or dzdydx.

$$\begin{aligned}
\int\int\int_{\mathbf{W}} x^2 dV &= \int_{-1}^{2}\int_{0}^{1}\int_{2y}^{4y} x^2 dzdydx \\
&= \int_{0}^{1}\int_{2y}^{4y}\int_{-1}^{2} x^2 dxdzdy \\
&= \int_{0}^{1}\int_{2y}^{4y} \frac{1}{3}x^3 \Big|_{-1}^{2} dzdy \\
&= \int_{0}^{1}\int_{2y}^{4y} \frac{9}{3} dzdy \\
&= \int_{0}^{1} 6ydy \\
&= 3y^2 \Big|_{0}^{1} \\
&= 3
\end{aligned}$$

5.a) Using spherical coordinates, find the equation of the cone surface $z = 2\sqrt{x^2+y^2}$.

We can use trigonometry to find the polar (azimuth) angle for the cone z=2r. This angle is constant for all points on the cone. For example when r = 1, z=2 and we can use a right angle triangle with hypotenuse $\sqrt{5}$ and polar angle ϕ to determine the value of this angle. In particular slope of the hypotenuse is 2, so

$$\tan(\phi) = \frac{1}{2}, \quad \phi = \arctan\left(\frac{1}{2}\right) = \arccos\left(\frac{2}{\sqrt{5}}\right)$$

b) Find the volume of the solid region **below** the cone $z = 2\sqrt{x^2+y^2}$, **above** the x-y plane, and **inside** the sphere $x^2+y^2+z^2 = 10$.

We notice that the sphere radius is $\sqrt{10}$ and make the following observation: it is **not** possible to integrate this region using cyliindrical coordinates (unless you are very careful with the bounds, and integrate in the r variable first $\frac{z}{2} \leq r\ \sqrt{10-z^2}$. However it is much easier to use spherical coordinattes.

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi}\int_{\arccos\left(\frac{2}{\sqrt{5}}\right)}^{\frac{\pi}{2}}\int_0^{\sqrt{10}} \rho^2\sin(\phi)d\rho d\phi d\theta \\
&= 2\pi\int_{\arccos\left(\frac{2}{\sqrt{5}}\right)}^{\frac{\pi}{2}}\int_0^{\sqrt{10}} \rho^2\sin(\phi)d\rho d\phi \\
&= 2\pi\int_{\arccos\left(\frac{2}{\sqrt{5}}\right)}^{\frac{\pi}{2}} \frac{1}{3}\rho^3\Big|_0^{\sqrt{10}}\sin(\phi)d\phi \\
&= \frac{2\pi}{3}\sqrt{1000}\left(-\cos(\phi)\Big|_{\arccos\left(\frac{2}{\sqrt{5}}\right)}^{\frac{\pi}{2}}\right) \\
&= \frac{2\pi}{3}\sqrt{1000}\left(\frac{2}{\sqrt{5}}\right)
\end{aligned}$$