# Solutions #11

**1.** Evaluate $\int_Q \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}}$ where $\vec{\boldsymbol{E}}(x,y,z) := ze^{x^2}\vec{\boldsymbol{\imath}} + 3y\vec{\boldsymbol{\jmath}} + (2 - yz^7)\vec{\boldsymbol{k}}$ and $Q$ is the union of the five "upper" faces of the unit cube $[0,1] \times [0,1] \times [0,1]$ orient outward. The face $z = 0$ is *not* part of $Q$.

*Solution.* We write $W := [0,1] \times [0,1] \times [0,1]$ for the unit cube. Let $P$ denote the square face with $z = 0$ oriented downward, so that $\partial W = Q + P$. The Divergence Theorem implies that

$$\int_P \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} + \int_Q \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} = \int_{\partial W} \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} = \int_W \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} \, dV$$

$$\iff \qquad \int_Q \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} = \int_W \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} \, dV - \int_P \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}}.$$

Moreover, we have

$$\begin{aligned}
\int_W \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{E}} \, dV &= \int_0^1 \int_0^1 \int_0^1 2xze^{x^2} + 3 - 7yz^6 \, dx \, dz \, dy = \int_0^1 \int_0^1 \left[ze^{x^2} + (3 - 7yz^6)x\right]_0^1 dz \, dy \\
&= \int_0^1 \int_0^1 ze + 3 - 7yz^6 - z \, dz \, dy = \int_0^1 \left[\tfrac{1}{2}(e-1)z^2 + 3z - yz^7\right]_0^1 dy \\
&= \int_0^1 \tfrac{1}{2}(e-1) + 3 - y \, dy = \left[\tfrac{1}{2}(e-1)y + 3y - \tfrac{1}{2}y^2\right]_0^1 = \tfrac{1}{2}e + 2 \\
\int_P \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} &= \int_P \vec{\boldsymbol{E}} \cdot (-\vec{\boldsymbol{k}}) \, dA = \int_P -2 \, dA = -2 \operatorname{Area}(P) = -2
\end{aligned}$$

so we deduce that $\int_Q \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} = \frac{1}{2}e + 4$. □

*Alternative Solution.* Parametrizing each of the five upper faces yields

$$\begin{aligned}
\int_Q \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{S}} &= \int_0^1 \int_0^1 \vec{\boldsymbol{E}}(1,y,z) \cdot \vec{\boldsymbol{\imath}} \, dy \, dz + \int_0^1 \int_0^1 \vec{\boldsymbol{E}}(0,y,z) \cdot (-\vec{\boldsymbol{\imath}}) \, dy \, dz \\
&\quad + \int_0^1 \int_0^1 \vec{\boldsymbol{E}}(x,1,z) \cdot \vec{\boldsymbol{\jmath}} \, dx \, dz + \int_0^1 \int_0^1 \vec{\boldsymbol{E}}(x,0,z) \cdot (-\vec{\boldsymbol{\jmath}}) \, dx \, dz \\
&\quad + \int_0^1 \int_0^1 \vec{\boldsymbol{E}}(x,y,1) \cdot \vec{\boldsymbol{k}} \, dx \, dy \\
&= \int_0^1 \int_0^1 ze - z \, dy \, dz + \int_0^1 \int_0^1 3 \, dx \, dz + \int_0^1 \int_0^1 2 - y \, dx \, dy \\
&= \left[\frac{1}{2}(e-1)z^2\right]_0^1 + 3 + \left[2y - \frac{1}{2}y^2\right]_0^1 = \frac{1}{2}e + 4\,.
\end{aligned}$$

□

**2.** Let $S$ be the surface defined by $z = e^{1-x^2-y^2}$ with $z \geqslant 1$ oriented upward and let $\vec{\boldsymbol{H}}(x,y,z) := x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + (2-2z)\vec{\boldsymbol{k}}$. Calculate $\int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}}$.

*Solution.* Let $W$ be the region bounded above by the surface $S$ and below by the upward oriented unit disk $D := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 \leqslant 1, z = 1\}$, so that $\partial W = S - D$. The

Divergence Theorem implies that

$$\int_{-D} \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} + \int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} = \int_{\partial W} \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} = \int_W \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{H}} \, dV = \int_W (1 + 1 - 2) \, dV = 0 \,,$$

whence $\int_D \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} = \int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}}$. Moreover, we have

$$\int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} = \int_D \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} = \int_D \vec{\boldsymbol{H}} \cdot \vec{\boldsymbol{k}} \, dA = \int_D 0 \, dA = 0 \,. \qquad \square$$

*Alternative Solution.* The surface $S$ is parametrized by $\vec{\boldsymbol{\sigma}} \colon [0,1] \times [0, 2\pi] \to \mathbb{R}^3$ where $\vec{\boldsymbol{\sigma}}(r, \theta) := r\cos(\theta)\vec{\boldsymbol{\imath}} + r\sin(\theta)\vec{\boldsymbol{\jmath}} + e^{1-r^2}\vec{\boldsymbol{k}}$. Since the normal vector is

$$\tfrac{\partial \vec{\boldsymbol{\sigma}}}{\partial r} \times \tfrac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta} = \det \begin{bmatrix} \vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}} \\ \cos(\theta) & \sin(\theta) & -2re^{1-r^2} \\ -r\sin(\theta) & r\cos(\theta) & 0 \end{bmatrix} = 2r^2e^{1-r^2}\cos(\theta)\vec{\boldsymbol{\imath}} + 2r^2e^{1-r^2}\sin(\theta)\vec{\boldsymbol{\jmath}} + r\vec{\boldsymbol{k}} \,,$$

we have

$$\begin{aligned}
&\int_S \vec{\boldsymbol{H}} \cdot d\vec{\boldsymbol{S}} \\
&= \int_0^1 \int_0^{2\pi} \vec{\boldsymbol{H}}\big(r\cos(\theta), r\sin(\theta), e^{1-r^2}\big) \cdot \big(2r^2e^{1-r^2}\cos(\theta)\vec{\boldsymbol{\imath}} + 2r^2e^{1-r^2}\sin(\theta)\vec{\boldsymbol{\jmath}} + r\vec{\boldsymbol{k}}\big) \, d\theta \, dr \\
&= \int_0^1 \int_0^{2\pi} 2r^3e^{1-r^2} + (2 - 2e^{1-r^2})r \, d\theta \, dr = 2\pi \int_0^1 2r^3e^{1-r^2} - 2re^{1-r^2} + 2r \, dr \\
&= 2\pi\big[-(1+r^2)e^{1-r^2} + e^{1-r^2} + r^2\big]_0^1 = 0 \,.
\end{aligned}$$

Observe that integration by parts, with $u = r^2$ and $dv = 2re^{1-r^2} \, dr$, gives

$$\int 2r^3e^{1-r^2} \, dr = -r^2e^{1-r^2} + \int 2re^{1-r^2} \, dr = -(1+r^2)e^{1-r^2} + C \,. \qquad \square$$

**3(a).** Consider a vector field $\vec{\boldsymbol{F}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $\vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}}(x,y,z) = x^2 + y^2 + 3$. Let $M$ be the boundary of a bounded solid oriented outwards. Can $\int_M \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}}$ be negative?

*Solution.* Let $W$ be the solid region such that $M = \partial W$. Since $x^2+y^2+3$ is everywhere positive, the Divergence Theorem implies that $\int_M \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{S}} = \int_W \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{F}} \, dV = \int_W x^2 + y^2 + 3 \, dV > 0$. $\square$

**3(b).** Find the flux of the vector field $\vec{\boldsymbol{G}}(x,y,z) = xy\vec{\boldsymbol{\imath}} + yz\vec{\boldsymbol{\jmath}} + zx\vec{\boldsymbol{k}}$ out of a sphere of radius 1 centered at the origin.

*Solution.* Let $B$ be the unit ball centered at the origin; its boundary is the sphere of radius 1 oriented outward. The Divergence Theorem implies that

$$\int_{\partial B} \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{S}} = \int_B \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{G}} \, dV = \int_B (y + z + x) \, dV \,.$$

Since the function $g \colon \mathbb{R}^3 \to \mathbb{R}$ given by $g(x,y,z) = x + y + z$ is symmetric about the origin $\big($which means that $g(-x,-y,-z) = -g(x,y,z)\big)$, the integral $\int_B g(x,y,z) \, dV$ is zero. $\square$

**Lemma.** *If $g(x,y,z)$ is symmetric about the origin and $B$ is the ball of radius $a$ centered at the origin, then $\int_B g(x,y,z)\,dV = 0$.*

*Proof.* The sphere of radius $a$ centered at the origin is given by $x^2+y^2+z^2=a^2$. Hence, we can express the triple integral as the following interated integral:

$$\int_B g(x,y,z)\,dV = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} g(x,y,z)\,dz\,dy\,dx\,.$$

If $u=-x$, $v=-y$ and $w=-z$ then the sphere is given by $u^2+v^2+w^2=a^2$, the Jacobian $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ equals $-1$ and we have

$$\begin{aligned}\int_B g(x,y,z)\,dV &= \int_{-a}^{a}\int_{-\sqrt{a^2-u^2}}^{\sqrt{a^2-u^2}}\int_{-\sqrt{a^2-u^2-v^2}}^{\sqrt{a^2-u^2-v^2}} g(-u,-v,-w)\left|\tfrac{\partial(x,y,z)}{\partial(u,v,w)}\right|\,dw\,dv\,du\\ &= \int_{-a}^{a}\int_{-\sqrt{a^2-u^2}}^{\sqrt{a^2-u^2}}\int_{-\sqrt{a^2-u^2-v^2}}^{\sqrt{a^2-u^2-v^2}} -g(u,v,w)\,dw\,dv\,du\\ &= -\left(\int_{-a}^{a}\int_{-\sqrt{a^2-u^2}}^{\sqrt{a^2-u^2}}\int_{-\sqrt{a^2-u^2-v^2}}^{\sqrt{a^2-u^2-v^2}} g(u,v,w)\,dw\,dv\,du\right) = -\int_B g(x,y,z)\,dV\,.\end{aligned}$$

Hence, $2\int_B g(x,y,z)\,dV = 0$ which implies that $\int_B g(x,y,z)\,dV = 0$. □

*Alternative Solution.* Converting to spherical coordinates, the triple integral becomes

$$\begin{aligned}\int_B x+y+z\,dV &= \int_0^1\int_0^{2\pi}\int_0^{\pi}\big(\rho\sin(\phi)\cos(\theta)+\rho\sin(\phi)\sin(\theta)+\rho\cos(\phi)\big)\rho^2\sin(\phi)\,d\phi d\theta d\rho\\ &= \left(\int_0^1 \rho^3 d\rho\right)\left[\left(\int_0^{2\pi}\cos(\theta)+\sin(\theta)d\theta\right)\left(\int_0^{\pi}\sin^2(\phi)d\phi\right)+2\pi\left(\int_0^{\pi}\sin(\phi)\cos(\phi)d\phi\right)\right]\\ &= \frac{1}{4}\left(\big[\sin(\theta)-\cos(\theta)\big]_0^{2\pi}\left[-\frac{1}{2}\sin(\phi)\cos(\phi)+\frac{1}{2}\phi\right]_0^{\pi}+2\pi\left[\frac{1}{2}\sin^2(\phi)\right]_0^{\pi}\right)\\ &= \frac{1}{4}\left[(0)\left(\frac{1}{2}\pi\right)+\pi(0)\right] = 0\,.\end{aligned}$$

□

*Another Solution.* The unit sphere $S$ centered is parametrized by $\vec{\boldsymbol{\sigma}}\colon [0,\pi]\times[0,2\pi]\to\mathbb{R}^3$ where $\vec{\boldsymbol{\sigma}}(\phi,\theta) := \sin(\phi)\cos(\theta)\vec{\boldsymbol{\imath}}+\sin(\phi)\sin(\theta)\vec{\boldsymbol{\jmath}}+\cos(\phi)\vec{\boldsymbol{k}}$. Since

$$\begin{aligned}\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial\phi}\times\frac{\partial\vec{\boldsymbol{\sigma}}}{\partial\theta} &= \det\begin{bmatrix}\vec{\boldsymbol{\imath}} & \vec{\boldsymbol{\jmath}} & \vec{\boldsymbol{k}}\\ \cos(\phi)\cos(\theta) & \cos(\phi)\sin(\theta) & -\sin(\phi)\\ -\sin(\phi)\sin(\theta) & \sin(\phi)\cos(\theta) & 0\end{bmatrix}\\ &= \sin^2(\phi)\cos(\theta)\vec{\boldsymbol{\imath}}+\sin^2(\phi)\sin(\theta)\vec{\boldsymbol{\jmath}}+\sin(\phi)\cos(\phi)\vec{\boldsymbol{k}}\end{aligned}$$

we have

$$\int_S \vec{\boldsymbol{G}} \cdot d\vec{\boldsymbol{S}} = \int_0^{\pi} \int_0^{2\pi} \vec{\boldsymbol{G}}\big(\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\big) \cdot \left(\tfrac{\partial \vec{\boldsymbol{\sigma}}}{\partial \phi} \times \tfrac{\partial \vec{\boldsymbol{\sigma}}}{\partial \theta}\right)\, d\theta\, d\phi$$

$$= \int_0^{\pi} \int_0^{2\pi} \sin^4(\phi)\cos^2(\theta)\sin(\theta) + \sin^3(\phi)\cos(\phi)\sin^2(\theta) + \sin^2(\phi)\cos^2(\phi)\cos(\theta)\, d\theta\, d\phi$$

$$= \left(\int_0^{\pi} \sin^4(\phi)\, d\phi\right)\left(\int_0^{2\pi} \cos^2(\theta)\sin(\theta)\, d\theta\right) + \left(\int_0^{\pi} \sin^3(\phi)\cos(\phi)\, d\phi\right)\left(\int_0^{2\pi} \sin^2(\theta)\, d\theta\right)$$

$$+ \left(\int_0^{\pi} \sin^2(\phi)\cos^2(\phi)\, d\phi\right)\left(\int_0^{2\pi} \cos(\theta)\, d\theta\right)$$

$$= \left(\int_0^{\pi} \sin^4(\phi)\, d\phi\right)\left[-\tfrac{1}{3}\cos^3(\theta)\right]_0^{2\pi} + \left[\tfrac{1}{4}\sin^4(\phi)\right]_0^{\pi}\left(\int_0^{2\pi} \sin^2(\theta)\, d\theta\right)$$

$$+ \left(\int_0^{\pi} \sin^2(\phi)\cos^2(\phi)\, d\phi\right)\left[\sin(\theta)\right]_0^{2\pi}$$

$$= \left(\int_0^{\pi} \sin^4(\phi)\, d\phi\right)(0) + (0)\left(\int_0^{2\pi} \sin^2(\theta)\, d\theta\right) + \left(\int_0^{\pi} \sin^2(\phi)\cos^2(\phi)\, d\phi\right)(0) = 0\,. \qquad \square$$