Math 227 Queen's University, Department of Mathematics

Vector Analysis, Homework 1-Solutions

September 2013

1. Question number 8, section 15.1 Stewart.

The midpoint rule we will use consists of taking the average of the **minimum=m** and the **maximum= M** value of the function $f(x, y)$ over each subrectangle in the domain. There are four equal subrectangles which we can label $A_{1,1}, A_{1,2}, A_{2,1}, A_{2,2}$. Each subrectangle $A_{i,j}$ having $\Delta A_{i,j} = 1$ is indexed with i denoting the row, and j denoting the column of the corresponding subrectangle.

The function in question is described using its level sets and we will use this information to determine the following table of values within the four subrectangles.

	m	**M**
$A_{1,1}$	1	3
$A_{1,2}$	1	6
$A_{2,1}$	2	5
$A_{2,2}$	3	7

To estimate $\int\int_{\mathbf{R}} f(x, y)dA$ we use four equal subintervals as indicated in the table above, and we average the minimum and maximum values in each subinterval to obtain

$$\int\int_{\mathbf{R}} f dA \simeq \sum_j \sum_i \frac{1}{2}(\mathbf{m}_{i,j} + \mathbf{M}_{i,j})\Delta A_{i,j} = (2 + 3.5 + 3.5 + 5) = 14$$

To improve this estimate we should subdivide the domain $\mathbf{R}$ into more subrectangles.

2. Question number 30, Section 15.2 Stewart

Volume in the first octant bounded by cylinder $z = 16 - x^2$ and the plane $y = 5$. This parabolic cylinder is parallel to the y-axis. In the first octant it lies over a rectanglular region

$$\mathbf{R} = \{(x, y) \mid 0 \le x \le 4, \quad 0 \le y \le 5\}$$

The crossections perpendicular to the x-axis are rectangles, and

$$\begin{aligned}
\text{Volume} &= \int\int_{\mathbf{R}} z dA \\
&= \int_0^4 A(x) dx \\
&= \int_0^4 5(16 - x^2) dx \\
&= 80x \Big|_0^4 - \frac{5}{3}x^3 \Big|_0^4 \\
&= 320 - \frac{320}{3} \\
&= \frac{640}{3}
\end{aligned}$$

If we crossection perpendicular to the y-axis we get parabolic curves bounding the crossection

$$\text{Volume} = \int\int_{\mathbf{R}} z dA$$

$$\begin{aligned}
&= \int_0^5 A(y)dy \\
&= \int_0^5 \int_0^4 (16 - x^2)dxdy \\
&= 80x \Big|_0^4 - \frac{5}{3}x^3 \Big|_0^4 \\
&= 320 - \frac{320}{3}
\end{aligned}$$

3. Question number 28, Section 15.3 Stewart. Use a double integral, and sketch the region in $\mathbf{R^3}$. There are two possible questions, depending on which edition 6 or 7 of Stewart you have. I will give both solutions.

Volume bounded by the cylinders $x^2 + y^2 = r^2$ and $y^2 + z^2 = r^2$.

The cylinder $x^2 + y^2 = r^2$ intersects the horizontal plane $z = 0$ in a circle of radius r, centered at the origin. Call the region interior to this circle **D**. The secret to unlocking this problem is to crossection this domain perpendicular to the y-axis. The crossections perpendicular to the x-axis give more complicated formulas.

Look at a crossection in the region **D** which is perpendicular to the y-axis. The cylinder $y^2 + z^2 = r^2$ intersects this vertical crossection in a square of sidelength $2\sqrt{r^2 - y^2}$. To see this notice that the x-variable is bounded on this crossection by $-\sqrt{r^2 - y^2} \leq x \leq \sqrt{r^2 - y^2}$. The vertical crossection above this line segment is parallel to the cylinder $y^2 + z^2 = r^2$, and on this crossection the height of this cylinder above the horizonal plane is $z = \pm\sqrt{r^2 - y^2}$. The difference of the z values is $2\sqrt{r^2 - y^2}$. This shows that the crossection perpendicular to the y-axis is a square of sidelength $2\sqrt{r^2 - y_2}$ and area $4(r^2 - y^2)$. Denote this area by $A(y)$.

Next we will integrate $A(y)$ to get the volume of the intersecting cylinders which is the same as integrating twice the height of the upper half of the horizontal cylinder as a double integral over the region **D**

$$
\begin{aligned}
\text{Volume} &= \int\int_{\mathbf{D}} 2z dA \\
&= \int_{-r}^{+r} A(y) dy \\
&= \int_{-r}^{+r} 4(r^2 - y^2) dy \\
&= 4r^2 y \Big|_{-r}^{+r} - \frac{4}{3} y^3 \Big|_{-r}^{+r} \\
&= 8r^3 - \frac{8}{3} r^3 \\
&= \frac{16}{3} r^3
\end{aligned}
$$

In 7^{ed} the questions reads: The volume bounded by the planes

$$z = 0, z = x, x + y = 2, y = x.$$

These four planes bound a finite region in $\mathbb{R}^3$. To see this notice that the planes y=x, and x+y+2 are vertical planes and of course z=0 is horizontal. The extra plane z=x bounds the region. To see this notice that if $x < 0$ then $z < 0$ on the plane $z = x$. Since this cant happen we notice next that therefore $y > 0$ since otherwise $y = x < 0$ on the plane $y = x$. Therefore the region lies in the first octant , below the plane $z = x$ and above the region **D** in the horizontal plane bounded by $x = 0, y = x, x + y = 2$.

This is a isosceles triangular shaped region, with the symmetric vertex at $(1, 1)$. We can crossection this region perpendicular to the y-axis.

$$\begin{aligned}
\text{Volume} &= \int\int_{\mathbf{D}} z dA \\
&= \int_0^1 A(y) dy \\
&= \int_0^1 \int_y^{2-y} x dx dy \\
&= \int_0^1 \frac{1}{2} x^2 \Big|_y^{2-y} dy \\
&= \int_0^1 \frac{1}{2} \left(4 - 4y + y^2 - y^2\right) dy \\
&= \frac{1}{2} \left(4y - 2y^2\right) \Big|_0^1 \\
&= 1
\end{aligned}$$