Math 227 Queen's University, Department of Mathematics

**Vector Analysis, Homework 2-solutions**

September 2013

**1.** Evaluate the double integral $\int\int_{\mathbf{D}} x\cos(y)dA$ where $\mathbf{D}$ is bounded by the lines $y = 0, x = 1$ and the curve $x = \sqrt{y}$. Evaluate this integral in two different ways by reversing the order of integration.

the region $\mathbf{D}$ in question lies to the right of the parabola $x = \sqrt{y}$ in the first quadrant, and bounded on the right by the vertical line x=1.

**Solution 1** Integrating first with respect to x, keeping y fixed and then with respect to y we find

$$\begin{aligned}
\int\int_{\mathbf{D}} x\cos(y)dA &= \int_0^1\int_{\sqrt{y}}^1 x\cos(y)dxdy \\
&= \int_0^1 \frac{1}{2}x^2\Big|_{\sqrt{y}}^1 \cos(y)dy \\
&= \frac{1}{2}\int_0^1 (1-y)\cos(y)dy \quad (u = (1-y), dv = \cos(y)dy) \\
&= \frac{1}{2}\left((1-y)\sin(y)\Big|_0^1 + \int_0^1 \sin(y)dy\right) \\
&= -\frac{1}{2}(\cos(1)-1)
\end{aligned}$$

where we used integration by parts in the fourth line.

**Solution2** Interchanging the order of integration

$$\int\int_{\mathbf{D}} x\cos(y)dA = \int_0^1\int_0^{x^2} x\cos(y)dydx$$

$$
\begin{aligned}
&= \int_0^1 +x\sin(x^2)dx \\
&= \frac{1}{2}\int_0^1 \sin(u) \\
&= -\frac{1}{2}\cos(u)\Big|_0^1 \\
&= -\frac{1}{2}(\cos(1)-1)
\end{aligned}
$$

**2.** Using triple integrals, find the volume of the solid **W** that lies under the plane $4x+6y-2z+15=0$ and above the region

$$
\mathbf{R} = \Big\{(x,y)|x^2+y^2 \le 2\Big\}
$$

For this problem we first integrate over the vertical direction z, between the horizonatal plane $z=0$, and the top plane which is the graph of the function

$$
z = f(x,y) = \frac{1}{2}(4x+6y+15)
$$

We then compute the signed volume by integrating over the circular domain **R** using polar coordinates. The outer boundary of this domain is a cirlce of radius $\sqrt{2}$.

$$
\begin{aligned}
\text{signed volume} &= \int\int\int_{\mathbf{W}} dV \\
&= \int\int_{\mathbf{R}}\int_0^{\frac{1}{2}(4x+6y+15)} dzdA \\
&= \int\int_{\mathbf{R}} \frac{1}{2}(4x+6y+15)\,dA \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} (4r\cos(\theta)+6r\sin(\theta)+15)\,rdrd\theta \\
&= \int_0^{2\pi}\int_0^{\sqrt{2}} (4\cos(\theta)+6\sin(\theta))\,r^2drd\theta + \int_0^{2\pi}\int_0^{2} (15)\,rdrd\theta
\end{aligned}
$$

$$= \int_0^{2\pi} (4\cos(\theta) + 6\sin(\theta))\, d\theta \cdot \int_0^{\sqrt{2}} r^2 dr + 15 \int_0^{2\pi} \int_0^{\sqrt{2}} r dr d\theta$$

$$= 0 + 15\pi 2$$

**3.** Using spherical coordinates or cylindrical coordinates calculate the volume of the ice cream cone shaped region **W** between the surfaces $\phi = \frac{\pi}{3}$ ($\phi$ is the polar angle) and the sphere $\rho = \sqrt{3}$.

The cone $\phi = \frac{\pi}{3}$ lies below the sphere $\rho = \sqrt{3}$ until they intersect along a curve on the sphere of constant lattitude. This curve of course corresponds to the polar angle $\phi = \frac{\pi}{3}$ . We calculate the volume of **W** in either spherical or cylinder coordinates.

$$\begin{aligned}
\text{signed volume} &= \int\int\int_{\mathbf{W}} dV \\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^{\sqrt{3}} \rho^2 \sin(\phi) d\rho d\phi d\theta \\
&= \frac{1}{3}\sqrt{27} \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \sin(\phi) d\phi d\theta \\
&= 2\pi\sqrt{3} \int_0^{\frac{\pi}{3}} \sin(\phi) d\phi \\
&= 2\pi\sqrt{3} \cos(\phi) \Big|_{\frac{\pi}{3}}^{0} \\
&= 2\pi\sqrt{3} \left(1 - \frac{1}{2}\right) = \pi\sqrt{3}
\end{aligned}$$

We can do this calculation using cylinder coordinates, however the algebra is much more complicated. For this we need to use the fact that the circle of intersection of the cone and the sphere will have height $z = \sqrt{3}\cos(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. The corresponding value of the radial variable r on this circle is $r = \sqrt{3}\sin(\frac{\pi}{3}) = \frac{3}{2}$.

Finally, we also need the equation of the cone and the sphere in cylinder coordinates: $z = \tan(\frac{\pi}{6}) = \sqrt{3}r$, $\quad z = \sqrt{3 - r^2}$.

$$\begin{aligned}
\text{signed volume} &= \int\int\int_{\mathbf{W}} dV \\
&= \int_0^{2\pi}\int_0^{\frac{3}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{3-r^2}} r dz dr d\theta \\
&= 2\pi \int_0^{\frac{3}{2}} \left(\sqrt{3-r^2} - \frac{r}{\sqrt{3}}\right) r dr \quad \left(u = 3 - r^2, -\frac{du}{2} = r dr\right) \\
&= 2\pi \left[\left(-\frac{1}{2}\right)\frac{2}{3}\left(3 - r^2\right)^{\frac{3}{2}} - \frac{1}{3\sqrt{3}} r^3\right]_0^{\frac{3}{2}} \\
&= 2\pi \left[\sqrt{3} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8}\right] \\
&= 2\pi\sqrt{3}\left(1 - \frac{1}{2}\right) = \pi\sqrt{3}
\end{aligned}$$

**4.** Consider the following donut shaped surface given by the equation

$$\left(\sqrt{x^2 + y^2} - a\right)^2 + z^2 = b^2, \quad a > b > 0$$

(a) Sketch the crossection of this surface with the planes $z = 0, z = \pm b$ and the planes $x = 0, y = 0$. Sketch the surface.

The crossection of this surface with the plane $z = 0$ consists of two concentric circles with radii $r = a - b, a + b$. When $z = \pm b$, it follows that $z^2 = b^2$ and accordingly, $r = a$ in polar coordinates.

The crossections of the surface with coordinate planes $x = 0$ or $y = 0$ consist of two vertical circles, in their respective coordinate planes of radii b, and centered at

$x = \pm a$ and $y = \pm a$.

It can be seen that this surface has two symmetric parts, above and below the plane $z = 0$. The upper part of this surface corresponds to taking $z = \sqrt{b^2 - \left(\sqrt{x^2+y^2} - a\right)^2} = \sqrt{b^2 - (r-a)^2}$ in polar coordinates. The symmetric lower part of this surface corresponds to taking the negative square root of this expression.

(b) By representing the top half of the surface as the graph of a function over a domain **D** in the x-y plane, determine the domain **D**.

As explained in the paragraph above, the surface of the donut consists of two symmetric halves which come together on the plane $z = 0$. The top half of the surface corresponds to the graph of the function $z = \sqrt{b^2 - \left(\sqrt{x^2+y^2} - a\right)^2}$ and the domain of this function corresponds to the annular region between the circles of radii $r = a - b, a + b$.

(c) Write a double integral for the surface area of this surface over the domain **D**. Using polar coordinates, evaluate this integral.

We use the formula developed in class

$$\text{surface area} = \int\int_{\mathbf{D}} \sqrt{1 + f_x^2 + f_y^2}\, dA$$

First we need to consider taking partial derivatives of the function with respect to x and y variables. To simplify the exposition, we can substitute the polar distance $r = \sqrt{x^2 + y^2}$ after performing the calculation for the derivatives. We also convert to polar coordinates at the final computation for the derivatives.

$$
\begin{aligned}
z &= \sqrt{b^2 - \left(\sqrt{x^2+y^2} - a\right)^2} \\
f_x &= \frac{\frac{1}{2}(-2)(r-a)}{\sqrt{b^2-(r-a)^2}}\frac{x}{r} \\
&= \frac{-(r-a)\cos(\theta)}{\sqrt{b^2-(r-a)^2}} \\
f_y &= \frac{-(r-a)\sin(\theta)}{\sqrt{b^2-(r-a)^2}} \\
\sqrt{1+f_x^2+f_y^2} &= \sqrt{1+\frac{(r-a)^2}{b^2-(r-a)^2}} \\
&= \sqrt{\frac{b^2}{b^2-(r-a)^2}}
\end{aligned}
$$

Now we can integrate this over the annular domain which we discribed earlier

$$
\begin{aligned}
\int\int_{\mathbf{D}} \sqrt{1+f_x^2+f_y^2}\,dA &= \int_0^{2\pi}\int_{a-b}^{a+b}\sqrt{\frac{b^2}{b^2-(r-a)^2}}r\,dr\,d\theta \quad \left(u = b^2-(r-a)^2, -\frac{1}{2}du = (r-a)dr\right) \\
&= \int_0^{2\pi}\int_{a-b}^{a+b}\sqrt{\frac{b^2}{b^2-(r-a)^2}}(r-a)\,dr\,d\theta + \int_0^{2\pi}\int_{a-b}^{a+b}\sqrt{\frac{b^2}{b^2-(r-a)^2}}a\,dr\,d\theta \\
&= 2\pi\left[\frac{-1}{2}2\sqrt{b^2(b^2-(r-a)^2)}\,\Big|_{a-b}^{a+b} + \int_{a-b}^{a+b}\sqrt{\frac{b^2}{b^2-(r-a)^2}}a\,dr\right] \\
&= 2\pi ab\int_{a-b}^{a+b}\sqrt{\frac{1}{b^2-(r-a)^2}}dr \quad u = (r-a), du = dr \\
&= 2\pi ab\int_{-b}^{+b}\sqrt{\frac{1}{b^2-u^2}}du
\end{aligned}
$$

This last integral can be evaluated with the help of a table of integrals.