Math 227 Queen's University, Department of Mathematics

Vector Analysis, Homeworkz 3-solutions

September 2013

1. Evaluate the triple integral $\int\int\int_{\mathbf{W}} \frac{dV}{\sqrt{x^2+y^2+z^2}}$ where $\mathbf{W}$ is the solid region between the upper hemispheres of two concentric spheres of radii $a < b$.

We use spherical coordinates due to the spherical symmetry of this problem. In this coordinate system we notice that $\rho = \sqrt{x^2+y^2+z^2}$.

$$\begin{aligned}
\int\int\int_{\mathbf{W}} \frac{dV}{\sqrt{x^2+y^2+z^2}} &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_a^b \frac{\rho^2\sin(\phi)}{\rho} d\rho d\phi d\theta \\
&= 2\pi\frac{1}{2}(b^2-a^2)(-\cos(\phi))\Big|_0^{\frac{\pi}{2}} \\
&= \pi(b^2-a^2)
\end{aligned}$$

2.Consider the vector field $\mathbf{F}(x,y) = (-2y, 2x)$.

a) . Show that the parameterized curve $(\cos(2t), \sin(2t))$ is a flow line of this vector field. Show that the family of parameterized curves $(a\cos(2t), a\sin(2t))$ are flow lines of $\mathbf{F}$. What curve does this parameterization describe?

$$\frac{d}{dt}(\cos(2t), \sin(2t)) = (-2\sin(2t), 2\cos(2t)) = \mathbf{F}(\cos(2t), \sin(2t))$$

and similarly, for any value of $a > 0$,

$$\frac{d}{dt}(a\cos(2t), a\sin(2t)) = (-2a\sin(2t), 2a\cos(2t)) = \mathbf{F}(a\cos(2t), a\sin(2t))$$

which shows that every counterclockwise oriented circle with angular speed $\frac{d\theta}{dt} = 2$ is a flow line of this vector field.

b. Find a vector field $\mathbf{G}(x, y)$ which is everywhere perpendicular to the field $\mathbf{F}$.

If $\mathbf{G}(x, y) = (x, y)$, then $\mathbf{G}(x, y) \cdot \mathbf{F}(x, y) = -yx + xy = 0$. Therefore $\mathbf{G}(x, y)$ is perpendicular to $\mathbf{F}$(x,y) at every point. The geometric picture for this calculation, $\mathbf{F}$ is a tangent vector field to concentric circles centered at the origin in $\mathbb{R}^2$ and $\mathbf{G}$ is a radial vector field, tangent to straight lilnes through the origin in $\mathbb{R}^2$.

c. Find the flow line of the vector field $\mathbf{G}$ which goes through the point $(x, y), \ \ x^2 + y^2 > 0$, at $t = 0$.

In order to find solutions of $\frac{d\vec{r}}{dt} = \mathbf{G}(\vec{r}(t))$ we need to solve the differential equations

$$\frac{dx}{dt} = x(t), \quad \frac{dy}{dt} = y(t)$$

and find the solution which satisfies $x(0) = x, \ \ y(0) = y$. This solution is exponential,

$$x(t) = xe^t, \quad y(t) = ye^t.$$

and describes the half line from the origin through (x,y). Notice that the speed of this solution increases with time t. If we think of x,y as parameters for this family of flow lines, then we have parameterized the entire family of flow lines in $\mathbb{R}^2$.

3. Using spherical coordinates, find the volume of the spherical cap: the solid region

$x^2 + y^2 + z^2 \leq 10$ which lies above the plane $z = 1$.

THe key to visualizing this problem is to understand how to compute the polar angle Φ at every point of the curve of intersection of the plane z=1, with the sphere $x^2 + y^2 + z^2 = 10$, and then to recognize that the polar angle ϕ of the region will be bounded by $0 \leq \phi \leq \Phi$.

Let us first calculate Φ. If you set up a diagram with the spherical cap on top of the plane z=1, we see that the curve of intersection is the circle of radius 3, $x^2 + y^2 + 1 = 10$. Every point along this curve of intersection has constant polar angle ,which can be computed using a right angle triangle with vertex at (0,0,0) and right angle at the point (0,0,1), and sidelength 3 opposite the polar angle Φ.

$$\Phi = \arctan(3) = \arcsin\left(\frac{3}{\sqrt{10}}\right)$$

Next we need to determine the spherical coordinates (ρ, θ, ϕ) of every point interior to the circle of intersection, on the plane z=1. Clearly the angle θ is unrestricted, $0 \leq \theta \leq 2\pi$. Moreover, the polar angle is restricted by our earlier computation $0 \leq \phi \leq \Phi$. This leaves only the distance from the origin ρ. This can be done using the relation from spherical to rectangular coordinates, $z = \rho\cos(\phi)$. Setting z=1 on the plane and solving for ρ gives the value for $\rho = \frac{1}{\cos(\phi)}$ for the point on the plane z=1.

We can now calculate the volume using the fact that the radial line from the origin through the plane z=1 and ending on the sphere $x^2 + y^2 + z^2 = 10$, corresponds to integrating in the polar distance ρ first of all.

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\begin{aligned}
\int\int\int_{\mathbf{W}} dV &= \int_0^{2\pi} \int_0^{\arcsin\left(\frac{3}{\sqrt{10}}\right)} \int_{\frac{1}{\cos(\phi)}}^{\sqrt{10}} \rho^2 \cos(\phi) d\rho d\phi d\theta \\
&= \frac{2\pi}{3} \int_0^{\arcsin\left(\frac{3}{\sqrt{10}}\right)} \rho^3 \cos(\phi) \Big|_{\frac{1}{\cos(\phi)}}^{\sqrt{10}} d\phi \\
&= \frac{2\pi}{3} \int_0^{\arcsin\left(\frac{3}{\sqrt{10}}\right)} \left[\sqrt{1000} - \frac{1}{\cos^3(\phi)}\right] \cos(\phi) d\phi \\
&= \frac{2\pi}{3} \int_0^{\arcsin\left(\frac{3}{\sqrt{10}}\right)} \left[\sqrt{1000}\cos(\phi) - \frac{1}{\cos^2(\phi)}\right] d\phi \\
&= \frac{2\pi}{3} \left[\sqrt{1000}\left(\frac{3}{\sqrt{10}}\right) - \tan(\phi)\Big|_0^{\arctan(3)}\right] \\
&= 27\frac{2\pi}{3}
\end{aligned}
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