Math 227 Queen's University, Department of Mathematics

Vector Analysis, Homeworkz 4-Solutions

October 2013

1. Consider the spiral curve which in polar coordinates is given by the equation $r = \theta$.

a) Find a vector parameterization for this curve in rectangluar coordinates.

The curve can be parameterized using polar coordinates, and setting the angle $\theta = t$, for time t. It follows that $\vec{r}(t) = (x(t), y(t))$, where

$$x(t) = r\cos(t), \quad y(t) = t\sin(t)$$

b) Compute the curvature at the points of the curve where $r = \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

We will use the formula for curvature using the arclength parameter s, the unit tangent vector $\vec{T}$, the velocity and acceleration vectors $\vec{r}' = \frac{d\vec{r}}{dt}, \vec{r}'' = \frac{d^2\vec{r}}{dt^2}$

$$\kappa(s) = \left\| \frac{d\vec{T}}{ds} \right\| = \frac{\|\vec{r}'' \times \vec{r}'\|}{\left\|\frac{d\vec{r}}{dt}\right\|^3}$$

First we compute the velocity vector, speed $\frac{ds}{dt}$, and the acceleration vector at arbitrary time t

$$\begin{aligned}
\frac{d\vec{r}}{dt} &= (\cos(t) - t\sin(t), \sin(t) + t\cos(t)) \\
\left\|\frac{d\vec{r}}{dt}\right\| &= \sqrt{1+t^2} \\
\frac{d^2\vec{r}}{dt^2} &= (-2\sin(t) - t\cos(t), 2\cos(t) - t\sin(t))
\end{aligned}$$

	$\vec{r}'$	$\vec{\mathbf{r}}''$	$\vec{\mathbf{r}}'' \times \vec{\mathbf{r}}'$	$\kappa(s)$
$\frac{\pi}{2}$	$(-\frac{\pi}{2}, 1)$	$(-2, -\frac{\pi}{2})$	$\left(-2 - \frac{\pi^2}{4}\right)\vec{k}$	$\frac{2+\frac{\pi^2}{4}}{\left(1+\frac{\pi^2}{4}\right)^{\frac{3}{2}}}$
π	$(-1, -\pi)$	$(\pi, -2)$	$(-2 - \pi^2)\vec{k}$	$\frac{2+\pi^2}{(1+\pi^2)^{\frac{3}{2}}}$
$\frac{3\pi}{2}$	$(\frac{3\pi}{2}, -1)$	$(2, \frac{3\pi}{2})$	$\left(-2 - \frac{9\pi^2}{4}\right)\vec{k}$	$\frac{2+\frac{9\pi^2}{4}}{\left(1+\frac{9\pi^2}{4}\right)^{\frac{3}{2}}}$

2. For the function $f(x, y) = 2x^2 + y^2$

a) Compute the family of flow lines for the gradient field $\vec{F}(x, y) = -\nabla f(x, y)$.

We begin by computing the negative gradient vector field for the potential function $f(x, y) = 2x^2 + y^2$.

$$-\nabla f(x, y) = -4x\vec{i} - 2y\vec{j}$$

For the lines of steepest descent we need to consider the differential equation $\frac{d\vec{r}(t)}{dt} = -\nabla f(\vec{r}(t)$ which in components amounts to the scalar differential equations and solutions (for arbitrary constants a,b)

$$\frac{dx}{dt} = -4x, \quad x(t) = ae^{-4t}, \quad \frac{dy}{dt} = -2y, \quad y(t) = be^{-2t}$$

b) Sketch some of the level curves of the function $f(x, y)$ together with some of the lines of steepest descent for this function.

To sketch the level curves and lines of steepest descent, we can first sketch the family of elliptic curves $2x^2 + y^2 = c^2$ which are ellipses which have semimajor axis along the y-axis and minor axis along the x-axis. The lines of steepest descent are parabolas, which open out along the x-axis. This can be seen by eliminating the time parameter t, between the flow lines. This amounts to the equations

$$\frac{x}{a} = \left(\frac{y}{b}\right)^2$$

which are indeed parabolas as descibed above. To sketch these curves we should include the orientation coming from the vector field $-\nabla f$. This orientation is always directed towards the origin in this example, along these curves, since the flow lines correspond to decreasing values of the function $f(x, y)$.

3. For the vector field $\vec{F} = x^2 i + xyj + 2zk$, and the curve which is the intersection of the cylinder $(x-1)^2 + y^2 = 2$ with the plane $x + y + z = 4$, calculate the work done in moving a particle one complete cycle of the curve, with counterclockwise orientation (as viewed from above the plane).

First we parameterize the circle $(x-1)^2+y^2$ in the plane $z = 0$. We use vector addition to conclude that the parameterized circle is $\vec{r_c}(t) = (1+\sqrt{2}\cos(t), \sqrt{2}\sin(t)),\ \ 0 \leq t \leq 2\pi$. Next we can see how the curve of intersection $\vec{r_i}(t)$ projects onto this circle. Namely the part of the plane lying over the circle $\vec{r_c}$ is the parameterized curve

$$\vec{r_i}(t) = \left(1 + \sqrt{2}\cos(t), \sqrt{2}\sin(t), 3 - \sqrt{2}\cos(t) - \sqrt{2}\sin(t)\right), \quad 0 \leq t \leq 2\pi$$

For this curve we have

$$\frac{d\vec{r_c}(t)}{dt} = \left(-\sqrt{2}\sin(t), \sqrt{2}\cos(t), \sqrt{2}\sin(t) - \sqrt{2}\cos(t)\right)$$

Along the curve $\vec{r}_i(t)$ we have

$$x^2(t) = \left(1+\sqrt{2}\cos(t)\right)^2, \;\; xy = \left(1+\sqrt{2}\cos(t)\right)\sqrt{2}\sin(t), \;\; z(t) = 3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)$$

$$\begin{aligned}
\int_C \vec{F}\cdot\vec{T}ds &= \int_0^{2\pi} \vec{F}\cdot\frac{d\vec{r}_i}{dt}dt \\
&= \int_0^{2\pi} \left(1+\sqrt{2}\cos(t)\right)^2\left(-\sqrt{2}\sin(t)\right)dt \\
&+ \int_0^{2\pi} \left(1+\sqrt{2}\cos(t)\right)\left(\sqrt{2}\sin(t)\right)\left(\sqrt{2}\cos(t)\right)dt \\
&+ \int_0^{2\pi} 2\left(3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)\right)\left(\sqrt{2}\sin(t)-\sqrt{2}\cos(t)\right)dt \\
&= 0
\end{aligned}$$

Each of the three integrals in the above calculation for work are independently 0. In the first two integrals, substitute $u = \sqrt{2}\cos(t), du = -\sqrt{2}\sin(t)dt$. For the last of the three integrals, substitute $u = \left(3-\sqrt{2}\cos(t)-\sqrt{2}\sin(t)\right), du = \left(\sqrt{2}\sin(t)-\sqrt{2}\cos(t)\right)dt$