Problem Set #12
Due: 7 December 2011

1. **(a)** Show that the path $\vec{\gamma}\colon [0, 2\pi] \to \mathbb{R}^3$ defined by $\vec{\gamma}(t) := \cos(t)\vec{\imath} + \sin(t)\vec{\jmath} + \sin(2t)\vec{k}$ lies on the surface $z = 2xy$.

(b) Evaluate $\int_C \big(y^3 + \cos(x)\big)\,dx + \big(\sin(y) + z^2\big)\,dy + x\,dz$ where C is the closed curve parametrized by $\vec{\gamma}$.

2. **(a)** Evaluate the circulation of the vector field $\vec{G}(x, y, z) := xy\vec{\imath} + z\vec{\jmath} + 3y\vec{k}$ around a square of side length 6, centered at the origin lying in the yz-plane, and oriented counterclockwise viewed from the positive x-axis.

(b) Let $\vec{H}(x, y, z) := (y - z)\vec{\imath} + (x + z)\vec{\jmath} + xy\vec{k}$ and let C be the circle of radius 3 centered at $(2, 1, 0)$ in the xy-plane oriented counterclockwise when viewed from above. Compute $\int_C \vec{H} \cdot d\vec{r}$. Is $\vec{H}$ path-independent? Explain.

3. Water in a bathtub has velocity vector field near the drain given, for x, y, z in cm, by

$$\vec{V}(x, y, z) := \frac{-y\vec{\imath} + x\vec{\jmath}}{(z^2+1)^2} + \frac{-z(x\vec{\imath} + y\vec{\jmath})}{(z^2+1)^2} - \frac{\vec{k}}{z^2+1} = -\frac{y + xz}{(z^2+1)^2}\vec{\imath} - \frac{yz - x}{(z^2+1)^2}\vec{\jmath} - \frac{1}{z^2+1}\vec{k}\ \text{cm}\cdot\text{s}^{-1}\,.$$

(a) The drain in the bathtub is a disk in the xy-plane with center at the origin and radius 1 cm. Find the rate at which the water is leaving the bathtub.

(b) Find the flux of the water through the hemisphere of radius 1, centered at the origin, lying below the xy-plane and oriented downward.

(c) Consider the vector field

$$\vec{U}(x, y, z) := \frac{1}{2}\left(\frac{y}{z^2+1}\vec{\imath} - \frac{x}{z^2+1}\vec{\jmath} - \frac{x^2 + y^2}{(z^2+1)^2}\vec{k}\right).$$

Compute $\int_E \vec{U} \cdot d\vec{r}$ where E is the edge of the drain oriented clockwise when viewed from above.

(d) Calculate $\vec{\nabla} \times \vec{U}$ and explain why your answers in parts (a) and (c) are equal.