Name and Student Number

MTHE 227, Vector Calculus, Fall 2014
Queen's University, Department of Mathematics
Please write your student number and your name clearly at the top of this page.

Additional space for calculations can be arranged on the back of each page or on additional blank pages at the end of the exam.
Do all five questions, marks are indicated. Total marks are 54.

1. [**6 marks**] The table gives values of the function $f(x, y)$. Use this table to find upper and lower Riemann sums for $\int\int_{\mathbf{R}} f dA$, over the rectangle

$$\mathbf{R} = \{(x, y) \,|\, 2 \leq x \leq 3, \quad -3 \leq y \leq -1\}$$

	x	2.0	2.5	3.0	
y					
-1.0		-4	0	-2	
-2.0		2	3	6	
-3.0		1	4	7	

There are four subrectangles, each having area $\Delta A = 0.5$. Taking the minimum $\mathbf{m}_{i,j}$ and the maximum $\mathbf{M}_{i,j}$ in each of these rectangles, gives the following

$$\text{Lower Riemann sum} = \frac{-4 - 2 + 1 + 3}{2} = -1$$
$$\text{Upper Riemann sum} = \frac{3 + 6 + 4 + 7}{2} = 10$$

2. For the vector field $\vec{\mathbf{F}}(x,y) = y\vec{\mathbf{i}} + 2x\vec{\mathbf{j}}$
a)[4 marks] Sketch the vector field $\vec{F}$ along the parabola $y = 4 - x^2$, between the points (3,-5) to (-2,0) .

Sketch the parabola, whose vertex lies at x=0, y=4, opening downwards. Sketch several values of the vector field (not worrying too much about the magnitude) along the parabola, for example at the endpoints, and the points where the parabola intersects the axes, $x = \pm 2, y = 0$, and x=0, y=4.

b) [8 marks] Find the work done by the vector field $\vec{F}$ around the counterclockwise oriented piecewise smooth closed curve which consists of $C_1 : y = 4 - x^2$ from (3,-5) to (-2,0), and the straight path C_2 from (-2,0) to (3,-5).

For the path C_1, use the parameterized path $\vec{r}_1(t) = (t, 4 - t^2), -2 \leq t \leq 3$. We will change the orientation of this parameterization to reflect the orientation of the path C_1 as we do the calculation. Notice that $\vec{r}_1' = (1, -2t)$.

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int_3^{-2} (4 - t^2, 2t) \cdot (1, -2t) dt \\ &= \int_3^{-2} 4 - 5t^2 dt \\ &= \frac{5}{3}[27 + 8] - 20 \\ &= \frac{155}{3} \end{aligned}$$

For the path C_2 use the parameterization $\vec{r}_2(t) = (t, -(t+2)), -2 \leq t \leq 3$. The tangent vector is $\vec{r}_2' = (1, -1)$.

$$\begin{aligned} \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_{-2}^{3} (-(t+2), 2t) \cdot (1, -1) dt \\ &= \int_{-2}^{3} -3t - 2 dt \\ &= -\frac{35}{2} \end{aligned}$$

The total work done by the force $\vec{F}$ around the closed curve $C_1 + C_2$ is the sum of the two above.

3.a)[6 marks] Using triple integrals, find the volume of the region lying above the x-y plane, inside the cylinder $x^2 + y^2 = 4$, and below the parabaloid $z = 4 + x^2 + y^2$.

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_)^2 \int_0^{1+r^2} r dz dr d\theta \\
&= 2\pi \int_0^2 \left(4 + r^2\right) r dr \\
&= 2\pi[8 + 4] \\
&= 24\pi
\end{aligned}$$

b)[6 marks] Evaluate the integral $\int\int\int_{\mathbf{R}} (x - y + z) dV$ where $\mathbf{R}$ is the region in the first (positive) octant below the plane $z = 1 - x - y$

$$\begin{aligned}
\int\int\int_{\mathbf{R}} (x - y + z) dV &= \int_)^1 \int_0^{1-x} \int_)^{1-x-y} (x - y + z) dz dy dx \\
&= \int_0^1 \int_0^{1-x} \left[(x - y)z + \frac{1}{2}z^2\right] \Big|_0^{1-x-y} dy dx \\
&= \int_0^1 \int_0^{1-x} \left(1 - x - y + 2xy + 2y^2\right) dy dx \\
&= \int_0^1 \left[1 - x - x(1 - x) - \frac{3}{2}(1 - x)^2 + \frac{2}{3}(1 - x)^3 + 2x(1 - x)\right] dx \\
&= 1 - \frac{1}{2} - \frac{1}{2} + \frac{1}{3} - \frac{1}{2} + \frac{2}{12} + 1 - \frac{2}{3} \\
&= \frac{1}{3}
\end{aligned}$$

(practice with adding fractions!)

4. [12 marks] Find the average value of the function $f(x,y) = (x^2+y^2)$ over the square whose vertices are (2,0), (0,2), (-2,0), (0,-2).
hint. Consider the change of variables $u = x+y, v = x-y$) in the x-y plane.

We have the general expression for average value

$$\text{average value} \times \text{ (area of square)} = \int\int_{\mathbf{S}} \left(x^2+y^2\right) dA$$

Next we notice that the area of the square region **S** in the x-y plane is $(2\sqrt{2})^2 = 8$.
Using the suggested change of variables to calculate the integral term, we first compute the inverse functions

$$x = \frac{1}{2}(u+v), \qquad y = \frac{1}{2}(u-v)$$

and the Jacobian determinant

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

We take the absolute value of this since it is negative. We then determine the limits of the new variables u,v and the corresponding domain in the uv-plane. These limits can be found by graphing the square **S** and noticing that the sides are given by the equations $u = \pm 2, v = \pm 2$. The limits on u,v therefore are

$$-2 \leq u \leq +2, \qquad -2 \leq v \leq +2$$

Using the change of variable formula from class, and calculating the integrand $(x^2+y^2) = \frac{1}{2}\left[u^2+v^2\right]$

$$\begin{aligned}
\int\int_{\mathbf{S}} \left(x^2+y^2\right) dA &= \int_{-2}^{+2}\int_{-2}^{+2} \frac{1}{2}\left[u^2+v^2\right]\frac{1}{2}dudv \\
&= \frac{1}{4}\int_{-2}^{+2}\left[u^2v + \frac{1}{3}v^3\right]_{v=-2}^{v=+2} du \\
&= \frac{1}{4}\int_{-2}^{+2} 4u^2 + \frac{16}{3}du \\
&= \left[\frac{1}{3}u^3 + \frac{4}{3}u\right]_{u=-2}^{u=+2} \\
&= \frac{16}{3} + \frac{16}{3} \\
&= \frac{32}{3}
\end{aligned}$$

Using the area of **S**=8, we calculate (average value) $= \frac{32}{24} = \frac{4}{3}$.

5.a)[4 marks] Using spherical coordinates in three dimensional space, find the polar (azimuth) angle ϕ, and the resulting equation in spherical coordinates of the cone surface $z = 3\sqrt{x^2 + y^2}$.

We can use trigonometry to find the polar (azimuth) angle for the cone z=3r. This angle is constant for all points on the cone. For example when r = 1, z=3 and we can use a right angle triangle with hypotenuse $\sqrt{10}$ and polar angle ϕ to determine the value of this angle. In particular slope of the hypotenuse is 3, so

$$\tan(\phi) = \frac{1}{3}, \quad \phi = \arctan\left(\frac{1}{2}\right) = \arccos\left(\frac{3}{\sqrt{10}}\right)$$

b) [6 marks] Find the volume of the solid region **below** the cone $z = 3\sqrt{x^2 + y^2}$, **above** the x-y plane, and **inside** the sphere $x^2 + y^2 + z^2 = 4$.

We notice that the sphere radius is 2 and make the following observation: it is **not** possible to integrate this region using cyliindrical coordinates (unless you are very careful with the bounds, and integrate in the r variable first . However it is much easier to use spherical coordinattes.

$$\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin(\phi) d\rho d\phi d\theta \\
&= 2\pi \int_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin(\phi) d\rho d\phi \\
&= 2\pi \int_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}} \frac{1}{3}\rho^3 \Big|_0^2 \sin(\phi) d\phi \\
&= \frac{16\pi}{3}\left(-\cos(\phi)\Big|_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}}\right) \\
&= 16\pi\left(\frac{1}{\sqrt{10}}\right)
\end{aligned}$$

c) [2 marks] Without doing the computation, set up a triple integral for the average value of the distance $\rho = \sqrt{x^2 + y^2 + z^2}$ over the solid region described in part **b)**.

This is an interesting computation.

$$\text{average value } \rho = \frac{\int_0^{2\pi} \int_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}} \int_0^2 \rho^3 \sin(\phi) d\rho d\phi d\theta}{\int_0^{2\pi} \int_{\arccos\left(\frac{3}{\sqrt{10}}\right)}^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin(\phi) d\rho d\phi d\theta}$$

Use the extra page at the back of the exam to write if necessary

Extra page for recording work and answers. Please indicate carefully which questions you are adding material here for.