Math 231, Introduction to Differential Equations, Fall 2011

Queen's University, Department of Mathematics

Homework 2, Due Thursday October 6

1. Consider the differential equation and intitial value

$$y' = 2ty^2, \quad y(a) = b$$

a) What is the largest rectangle in the $t - y$ plane for which the conclusions of the existence-uniqueness theorem are applicable.

b) Find the general solution, then compute the solution of the intial value problem $y(0) = b$ keeping the initial values as a parameter in the solution.

c) When $a = 0$, and $b > 0$,what is the interval of existence for $y(t, a, b)$? What happens to the solution as t approaches the endpoints of its interval of existence. Is this consistent with the conclusions of the existence uniqueness theorem?

2. Consider the differential equation

$$y' = f(y) = y(y - 1)(y - 2), \quad y(0) = y_0.$$

a) Sketch the graph of the slope function $f(y)$ versus y

b) Determine the equilibrium values of the differential equation $y' = f(y)$

c) Classify each equilibrium as stable or unstable. Sketch a few representative graphs of solutions in the $t - y$ plane

d) Can you explain what happens on the phase line when we change the slope function, by adding a parameter $f(y) + a$. By increasing the parameter a, find the value a_0 when two of the equilibria come together.

3. Consider the parameterized differential equation (which is used in the analysis of stability for fluid flow)

$$y' = \epsilon y - \sigma y^3, \quad \epsilon > 0, \quad \sigma > 0.$$

a) Solve this equation using the substitution $v = y^{-2}$, and write the solution in terms of its inital value $y(0) = y_0$. Using this formula show that every solution with initial value $y_0 > 0$ converges to an asymptotic value as $t \to +\infty$. Find this asymptotic value.

b) Find the equilibrium points and determine their stability. Put this information on a phase line diagram for this problem.

c) What happens as $\epsilon \to 0$? Draw a bifurcation diagram in the $\epsilon - y$ plane and indicate how the number of equilibrium points and their stability changes, as ϵ passes through 0, from negative to positive values.