Math 231, Introduction to Differential Equations, Fall 2011

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Homework 7, Not to hand in, Solutions

1) For each system, verify that (0,0) is an equilibrium point, show that the system is almost linear in a neighborhood, and discuss the type and stablility, by examining the linear part. Sketch the trajectories in a neighborhood of origin.

a) $\frac{dx}{dt} = x - y^2, \quad \frac{dy}{dt} = x - 2y + x^2$

b) $\frac{dx}{dt} = -x + y + 2xy, \quad \frac{dy}{dt} = -4x - y + x^2 - y^2$

a $f(x, y) = x - y^2$ which is already in the form of a Taylor expansion in powers of $(x - 0, y - 0)$. Similarly the linear part at $(0, 0)$ of $g(x, y)$ is $x - 2y$. The differential equation can be written

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}$$

The nonlinear part satisfies

$$G(x, y) = \begin{pmatrix} -y^2 \\ x^2 \end{pmatrix}, \quad \frac{\|G(x, y)\|}{\|(x, y)\|\,|} \to 0$$

which means that the system is almost linear at the equilibrium point $(0, 0)$. The theorem we studied in class tells us that the nonlinear system and the linear system have the same local behaviour provided the eigenvalues of the coefficient matrix have non 0 real parts. The eigenvalues are computed are the same as the diagonal elements of the matrix (it is upper triangular) which are $\lambda_1 = 1, \lambda_2 = -2$. The origin is an unstable saddle for both the linear and the nonlinear systems. The eigenvectors are $\xi_1 = (3, 1)^T$ and $\xi_2 = (0, 1)^T$. The eigenvector ξ_2 is the stable direction and ξ_1 is the unstable direction. The diagram will show hyperbolas coming in the stable direction, and going out along the unstable direction.

b $f(x, y) = -x + y + 2xy$ which is already in the form of a Taylor expansion in powers of $(x - 0, y - 0)$. Similarly the linear part at $(0, 0)$ of $g(x, y)$ is $-4x - y$. The differential equation can be written

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ -4 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}$$

The nonlinear part satisfies

$$G(x, y) = \begin{pmatrix} 2xy \\ x^2 - y^2 \end{pmatrix}, \quad \frac{\|G(x, y)\|}{\|(x, y)\|\,|} \to 0$$

which means that the system is almost linear at the equilibrium point $(0, 0)$. The theorem we studied in class tells us that the nonlinear system and the linear system have the same local behaviour provided the eigenvalues of the coefficient matrix have non 0 real parts. The eigenvalues are 0's of the characteristic polynomial $(-1 - \lambda)^2 +$

$4 = 0$. The eigenvalues are complex, $\lambda = -1 \pm i2$. The real parts are non 0. The origin is a stable focus (spiral point) for both the linear and the nonlinear systems. The complex eigenvector is $\xi_1 = (1, 2i)^T = (1, 0)^T + i(0, 2)^T$ The spiral point is rotating clockwise near the origin for both the linear and the nonlinear system.

2) For the following planar system

$$\text{DE} \quad \frac{dx}{dt} = 2x^2y - 3x^2 - 4y, \quad \frac{dy}{dt} = -2xy^2 + 6xy$$

a) Determine all equilibrium points.

The equilibrium points are determined by the simultaneous equations

$$y(2x^2 - 4) = 3x^2, \quad xy(-2y + 6) = 0$$

The second of the equations is easier to analyse, so either $x = 0, y = 0, y = 3$. Substituting these values into the first equation we get the three equilibrium points

$$(0, 0), \quad (-2, 3), \quad (2, 3)$$

b) By reparameterizing the trajectories with x, find a conservation law $H(x, y)$ for the trajectories of this system (DE).

Eliminating the time parameter amounts to reparameterising the solution curves with respect to x

$$\frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{dy}{dx} = \frac{-2xy^2 + 6xy}{y(2x^2 - 4) - 3x^2)}$$

Rewriting this in terms of a differential form

$$(2xy^2 - 6xy)dx + (y(2x^2 - 4) - 3x^2)dy = 0$$

Checking the condition for exactness we find that $M_y = 4xy - 6x = N_x$. The equation is exact and can be directly integrated.

$$F(x,y) = \int N(x,y)dy = \frac{1}{2}y^2(2x^2-4) - 3x^2y + h(x)$$

Computing the partial with respect to x for this function and equating this with $M(x,y)$

$$\frac{\partial F}{\partial x} = 2xy^2 - 6xy + h'(x) = 2xy^2 - 6xy$$

It follows that $h'(x) = 0$ and the implicit solution of the reparameterised equation is

$$F(x,y) = \frac{1}{2}y^2(2x^2-4) - 3x^2y = C$$

This is the conservation law for the solutions of the original differential equation. That is the function $F(x,y)$ is constant along the trajectories.

c) Classify the type and location of critical points which the function $H(x,y)$ has.

We can try to see what the phase portrait of the original system is by studying the critical points of the function $F(x,y)$. These come from the simultaneous equations

$$\frac{\partial F}{\partial x} = 2xy^2 - 6xy = 0,$$

$$\frac{\partial F}{\partial y} = y(2x^2-4) - 3x^2 = 0.$$

We have already analysed these solutions, there are three critical points, corresponding to the equilibrium points $(0,0), (\pm 2, 3)$.

To see what type these critical points are , we must use the second derivative test for functions of two variables. This requires that we compute the discriminant $D = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$ for each critical point.

$$\frac{\partial^2 F}{\partial x^2} = 2y^2 - 6y, \quad \frac{\partial^2 F}{\partial y^2} = 2x^2 - 4, \quad \frac{\partial^2 F}{\partial x \partial y} = 4xy - 6x$$

We can see from calculating the discriminant D at each critical point that $(0, 0)$ is indeterminant (D=0) while both critical points $(\pm 2, 3)$ are saddles ($D < 0$).

d) Show that the system (DE) is almost linear in the neighborhood of each critical point, and determine the linear part at each critical point.

The linear part at each critical point is obtained by calculating the partial derivatives at each equilibrium. At the equilibrium (0,0)

$$f_x(0, 0) = 0, \quad f_y(0, 0) = -4, \quad g_x(0, 0) = 0, g_y(0, 0) = 0$$

The system is almost linear at (0,0) but the linear part has a 0 eigenvalue, and we cannot conclude from the linear part what the nature of the equilibrium point is, which is also reflected in the fact that the critical point at (0,0) is indeterminate. Next the equilibrium at $(\pm 2, 3)$ are both saddles as can be ascertained by calculating the linear part at each point. The system**e)** Determine the eigenvalues and eigenvectors of the linear part at each critical point.

f) Using all the information you have determined from above, sketch the phase plane of trajectories of the system (DE).