Department of Mathematics and Statistics

Mathematics 231

Professor Daniel Offin

Midterm Examination-Solutions Fall 2011

Do any four of the following five questions

1. [12 marks] **a)** Find an integrating factor for the linear differential equation. **b)** Find the solution to the initial value problem. **c)** Indicate clearly on what interval this solution exists

$$\mathbf{(t-1)\frac{dy}{dt} - 5y = 3t, \quad y(2) = 3.}$$

a) first divide the equation by $(t-1)$ to put into standard form, then consider the

equation for the integrating factor

$$u'(t) = \frac{-5}{t-1}u(t), \quad \ln(u) = \int \frac{-5}{t-1}dt = -5\ln(t-1) = \ln(t-1)^{-5}$$

The integrating factor can be found by exponentiation of this equation,

$$u(t) = (t-1)^{-5}$$

b) Multiplying by the integrating factor we found in part a, we obtain the simpler differential equation to solve (using integration by parts for example)

$$\frac{d}{dt}\left[y(t)(t-1)^{-5}\right] = \frac{3t}{(t-1)^{-6}}, \; y(t)(t-1)^{-5} = \int \frac{3t}{(t-1)^{-6}}dt = -\frac{3t}{5(t-1)^5} - \frac{3}{20(t-1)^4} + C$$

multiplying by $(t-1)^5$ we get

$$y(t) = -\frac{3}{5} - \frac{3}{20}(t-1) + C(t-1)^5$$

Evaluating the initial conditions tells us that

$$3 = -\frac{3}{5} - \frac{3}{20} + C, \quad C = 3 + \frac{3}{5} + \frac{3}{20}$$

c) the interval of existence of this initial value problem should coincide with the interval on which the coefficients are continuous (after dividing by $(t-1)$) and which contain the value $t_0 = 2$. This interval is $1 < t < +\infty$.

2. [12 marks] Consider the parameterized differential equation with parameter $a \in \mathbb{R}$

$$\mathbf{y' = y(a - y^2)}$$

a) Find all equilibrium points as a function of the parameter a. **b)** Draw the bifurcation diagram (indicating the stable and unstable equilibria) and find the value a_0 where a bifurcation occurs.
a) The equilbrium values y_0 as a funtion of the parameter a, with corresponding parameter domain,

$$y_0 = 0, \quad a \in \mathbb{R}, \quad y_0 = \pm\sqrt{a}, \quad a \geq 0$$

The value $y_0 = 0$ is stable for $a \leq 0$ and unstable for $a > 0$. The values $y_0 = \pm\sqrt{a}$ are stable. The bifurcation value is $a_0 = 0$ where the two branches of stable equilibrium come together at $y_0 = 0$.
The bifurcation diagram shows a pitchfork change in the structure of the equilibrium points, with stable branches bifurcating out of the stable branch at $y_0 = 0$ (which after the bifurcation becomes unstable).

3. [12 marks] Solve the initial value problem, and sketch the graph of the solution on the $t - y$ plane, indicating clearly the zeroes of this solution.

$$\mathbf{y'' + 2y' - \frac{21}{4}y = 0, \quad y(0) = 0, \quad y'(0) = 2}$$

The characteristic equation for this polynomial operator is $r^2 + 2r - \frac{21}{4} = 0$. The roots of this polynomial by the quadratic formula are $r = -1 \pm \frac{1}{2}\sqrt{25} = -\frac{7}{2}, \frac{3}{2}$. The corresponding exponential solutions are $y_1 = e^{-\frac{7}{2}t}, y_2 = e^{\frac{3}{2}t}$, which are linearly independent over the real line $t \in \mathbb{R}$. Taking linear combinations of this fundamental set of solutions gives

$$y(t) = c_1 e^{-\frac{7}{2}t} + c_2 e^{\frac{3}{2}t}$$

Satisfying the initial conditions give us the linear equations

$$c_1 + c_2 = 0, \quad -\frac{7}{2}c_1 + \frac{3}{2}c_2 = 2, \quad c_1 = -\frac{2}{5}, c_2 = \frac{2}{5}$$

The solution of the initial value problem is

$$y(t) = -\frac{2}{5}e^{-\frac{7}{2}t} + \frac{2}{5}e^{\frac{3}{2}t}$$

This function is increasing for all $t \in \mathbb{R}$ since $y' > 0$. Therefore there is a single 0 value which occurs at $t = 0$.

4. [12 marks] Find three linearly independent solutions of the differential equation, and explain (show) why they are linearly independent and on which interval of time they are linearly independent.[Hint: (D+1) is a factor of the differential operator].

$$\mathbf{y''' + y'' - 4y' - 4y = 0}$$

The polynomial differential operator factors like $P(D) = (D+1)(D^2-4) = (D+1)(D-2)(D+2)$. The corresponding exponential solutions are $y_1 = e^{-t}, y_2 = e^{-2t}, y_3 = e^{2t}$. To show that they are linearly independent on the entire real line, we need only compute the Wronskian determinant at one point to see if it 0 or not 0.

$$W[y_1, y_2, y_3](0) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 2 \\ 1 & 4 & 4 \end{vmatrix} = -12$$

The solutions y_1, y_2, y_3 form a fundamental set of solutions on the entire real line (by Abel's theorem).

5. [12 marks] The following equation is not exact, but can be made exact with an integrating factor of the form $u = u(x)$. **a)** Find this integrating factor. **b)** Find the general solution of the differential equation

$$\mathbf{(4x + 2e^x)\, ydx + \left(3x^2 + 3e^x\right) dy = 0.}$$

a) Multiply the entire differential equation by $U(x)$ and apply the condition of exactness

$$U(x)(4x + 2e^x) = U'(x)\left(3x^2 + 3e^x\right) + U(x)\left(6x + 3e^x\right)$$

simplifying and solving for $U'(x)$ we get

$$U'(x) = U(x)\frac{-2x - e^x}{3x^2 + 3e^x}$$

Separating variables, and integrating we find

$$\ln(U(x)) = \int \frac{-2x - e^x}{3x^2 + 3e^x}dx = -\frac{1}{3}\ln(3x^2 + 3e^x)$$

From which it follows that

$$U(x) = (3x^2 + 3e^x)^{\frac{-1}{3}}$$

b Multiply the differential equation by this factor, we can integrate the term $N(x, y)$ to get

$$F(x, y) = \int (3x^2 + 3e^x)^{\frac{-1}{3}}\left(3x^2 + 3e^x\right)dy = (3x^2 + 3e^x)^{\frac{2}{3}}y + h(x)$$

We can check that $h'(x) = 0$ and we get the general solution

$$(3x^2 + 3e^x)^{\frac{2}{3}}y = C$$