Mathematics 231
Introduction to differential equations, Fall, 2009
Solutions Homework 1

1. The equation is linear with $p(t) = \cos(t)$,
so we multiply through by the integrating factor $e^{\sin(t)}$.

$$\begin{aligned}
y'e^{\sin(t)} + y\cos(t)e^{\sin(t)} &= 4\cos(t)e^{\sin(t)} \\
\frac{d}{dt}\left(ye^{\sin(t)}\right) &= 4\cos(t)e^{\sin(t)} \\
ye^{\sin(t)} &= \int 4\cos(t)e^{\sin(t)}dt \\
ye^{\sin(t)} &= 4e^{\sin(t)} + C \\
y &= 4 + Ce^{-\sin(t)} \qquad (1)
\end{aligned}$$

This is the general solution. To get the specific solution which satisfies the initial conditions, we substitute $t = \pi, \quad y = -1$ to find $C = -5$ This gives $y = 4 - 5e^{-\sin(t)}$.

2. a) The right hand side of the differential equation can be seen to be homogeneous, by dividing through numerator and denominator by x^2 we get

$$f(x,y) = \frac{1 + 3\left(\frac{y}{x}\right)^2}{2\left(\frac{y}{x}\right)} = g\left(\frac{y}{x}\right) = \frac{1}{2v} + \frac{3v}{2}, \quad v = \frac{y}{x}.$$

b) From the relation $xv = y$ we find on differentiation, that $x\frac{dv}{dx} + v = \frac{dy}{dx} = g(v)$. Solving for $\frac{dv}{dx}$ and using the expression we have for $g(v)$ from part a) we find

$$\frac{dv}{dx} = \frac{g(v) - v}{x} = \frac{1}{x}\left(\frac{1}{2v} + \frac{v}{2}\right) = \frac{v^2+1}{2vx}$$

$$\begin{aligned}
\frac{2vdv}{v^2+1} &= \frac{dx}{x} \\
\ln\left(v^2+1\right) &= \ln(x) + c \\
v^2 + 1 &= kx, \quad k > 0.
\end{aligned}$$

To graph this family, without solving for v, look at them as a family of parabolas over the v-axis. They are parabolas whose vertex lies on the positive x-axis, and which open out on the positive x axis. This family foliates the right half x-v plane, which means that they dont cross. The only vertical tangencies occur at the vertex of each parabola, which lies on the x-axis.

c) From the equation we found in part b), $\frac{y^2}{x^2} + 1 = kx$. Substituting the intial condition $y(1) = -1$ gives the constant $k = 2$. Solving

$$y^2 = 2x^3 - x^2, \quad y = -\sqrt{2x^3 - x^2}$$

which is defined on the interval of existence for this solution $x \geq \frac{1}{2}$. The vertical tangency occurs at $x = \frac{1}{2}$.

3. a). The model for growth of investment $S(t)$ at r percent interest per annum, compounded continuously is

$$\frac{dS}{dt} = \frac{r}{100}S + k \text{ Dollars per year}, \quad S(0) = 0$$

where k is the annual rate of investment required. This is a linear equation with integrating factor $e^{-0.075t}$ (assuming interest rate 7.5 percent per annum),

$$\begin{aligned}
e^{-.075t}S' - 0.075e^{-.075t}S &= ke^{-.075t} \\
\frac{d}{dt}\left(e^{-.075t}S\right) &= ke^{-.075t} \\
e^{-.075t}S &= \frac{-k}{0.075}e^{-.075t} + C \\
S &= -\frac{k}{0.075} + Ce^{.075t} \\
C &= \frac{k}{0.075} \\
S(t) &= \frac{k}{0.075}\left(e^{.075t} - 1\right) \qquad (2)
\end{aligned}$$

b) Using the value for r as above, and solving the last equation for k, when $S(40) = 10^6$, we find $40 \times 0.075 = 3$ so that

$$75 \times 10^3 = k\left(e^3 - 1\right), \quad k = 3930.81 \text{ dollars per year}$$

c) From part a) using a general interest rate r (percentage), we find

$$S(t) = \frac{100k}{r}\left(e^{\frac{r}{100}t} - 1\right)$$

Using the information $S(40) = 10^6, \quad k = 2000$, we find after simplification

$$10^6 = \frac{200,000}{r}\left(e^{0.4r} - 1\right), \quad 5r = e^{0.4r} - 1$$

This equation cannot be solved exactly, but a simple graphical solution using the second of the equations above is enough to convince us that such an interest rate r exists uniquely. Notice that both graphs $5r$ and $e^{0.4r} - 1$ have the value 0 when $r = 0$, but the slope of the linear graph is 5 while the slope of the exponential graph at 0 is 0.4. Since the exponential funtion (on the right hand side) grows more than linearly, while the left hand side grows linearly, the two graphs must cross at a unique positive value of r_0, which can then be determined numerically. A few calculations reveals that $9.7 < r_0 < 10$.