Mathematics 231
Introduction to differential equations, Fall, 2011
Solutions   Homework 3

1 Consider the differential equation and intitial value problem

$$bxydx + \big(3x^2 + 4\cos(y)\sin(y)\big)\, dy = 0, \quad y(1) = \frac{\pi}{4}$$

a) It is easy to see that the only value which makes this equation exact is b=6.
b) The equation is exact, since

$$\frac{\partial 6xy}{\partial y} = 6x = \frac{\partial\left(3x^2 + 4\cos(y)\sin(y)\right)}{\partial x}$$

We look for the general solution as the level set of some function $F(x, y)$

$$\begin{aligned}
F(x,y) &= \int 6xydx + h(y) \\
&= 3x^2y + h(y) \\
h'(y) &= \big(3x^2 + 4\cos(y)\sin(y)\big) - 3x^2 \\
&= 2\sin(2y) \\
h(y) &= -\cos(2y) \\
F(x,y) &= 3x^2y - \cos(2y) \\
F(1, \frac{\pi}{4}) &= \frac{3\pi}{4} \qquad (1)
\end{aligned}$$

The unique solution therefore coincides with the level set $3x^2y - \cos(2y) = \frac{3\pi}{4}$.

c) Some information on the interval of existence can be deduced from the differential equation itself, but of very limited nature. Without actually graphing some part of the level set which contains the intitial condition it is difficult to determine the exact nature of the interval of existence.
We rewrite the differential equation

$$\frac{dy}{dx} = \frac{-6xy}{(3x^2 + 4\cos(y)\sin(y))} = f(x, y)$$

$$\frac{\partial f}{\partial y} = \frac{-6x\left(3x^2 + 4\cos(y)\sin(y)\right) + 24xy\cos(2y)}{\left(3x^2 + 4\cos(y)\sin(y)\right)^2}$$

The denominator of both $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous functions which do not vanish for the pair of initial values $x = 1, y = \frac{\pi}{4}$. Therefore they dont vanish in some rectangle in the $x - y$ plane which contains this point, and by the existence uniqueness theorem, the initial value problem has a unique solution in *some* interval containing $x = 1$.

2. a) Multiply the equation by $h(x)$, and check the condition for exactness

$$h(x)\left(3yx^2+2xy+y^3\right)dx+h(x)\left(x^2+y^2\right)dy=0$$

$$\frac{\partial}{\partial y}h(x)\left(3yx^2+2xy+y^3\right)=\frac{\partial}{\partial x}h(x)\left(x^2+y^2\right)$$
$$h(x)\left(3x^2+2x+3y^2\right)=h'(x)\left(x^2+y^2\right)+h(x)2x$$

$$\begin{aligned} 3h(x)\left(x^2+y^2\right) &= h'(x)\left(x^2+y^2\right) \\ 3h(x) &= h'(x) \\ h(x) &= e^{3x} \text{ (now multiply equation by} e^{3x}) \\ e^{3x}\left(3yx^2+2xy+y^3\right)dx+e^{3x}\left(x^2+y^2\right) &= 0 \qquad (2) \end{aligned}$$

which is exact as can be checked by usual method.

b) We construct the function $F(x,y)$ which satisfies

$$\frac{\partial F}{\partial x}=e^{3x}\left(3yx^2+2xy+y^3\right),\quad \frac{\partial F}{\partial y}=e^{3x}\left(x^2+y^2\right)$$

This can be done by integrating first with respect to x or with respect to y. I will integrate with respect to y since this looks much easier....

$$\begin{aligned} F(x,y) &= \int e^{3x}\left(x^2+y^2\right)dy+v(x) \\ &= e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+v(x) \text{ (now differentiate with x)} \\ \frac{\partial F}{\partial x} &= 3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy+v'(x) \text{ (now replace } \frac{\partial F}{\partial x}) \end{aligned} \qquad (4)$$

$$e^{3x}\left(3x^2y+2xy+y^3\right)=3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy+v'(x)$$

from which we can now observe that $v'(x)=0$. Thus we have the general solution

$$F(x,y)=e^{3x}\left(x^2y+\frac{1}{3}y^3\right)=C$$

c) To construct the critical points of the function $F(x,y)$, we need consider the simulaneous equations

$$\frac{\partial F}{\partial x}=3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy=0$$

$$\frac{\partial F}{\partial y} = e^{3x}\left(x^2+y^2\right) = 0$$

The second equation clearly has the only solution $x = y = 0$ which is also a solution of the first equation. Therefore, there is a single critical point of F which is at the origin. The level curves of F can be seen using the Maple command contourplot, or we can apply the second derivative test to see what nature the critical point has. The level curves can be sketched from this information as well. For this we must calculate the discriminant

$$D = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$

In this example $D < 0$ which indicates that the origin is a saddle crtical point with two branches of the 0 level curve intersecting at $(0,0)$ and the contour lines nearby looking like pieces of hyperbolas.

3) Consider the coupled pair of first order equations

$$\frac{dx}{dt} = y^2 - x^2, \quad \frac{dy}{dt} = -2xy$$

a) If $q = x(t) + iy(t)$, where $i^2 = -1$, then

$$\frac{dq}{dt} = \frac{dx}{dt} + i\frac{dy}{dt} = (y^2 - x^2) - i2xy = -q^2$$

This equation can be integrated easily since it is separable $\frac{1}{q} = t + C$. To find $x(t), y(t)$ we need to find the real and imaginary parts of the complex function $q(t)$. We take recirocals and let $c = a + ib$ a complex constant of integration.

$$q(t) = \frac{1}{t+a+ib} = \frac{t+a-ib}{(t+a)^2+b^2} = \frac{t+a}{(t+a)^2+b^2} - i\frac{b}{(t+a)^2+b^2}$$

It follows that $x(t) = \frac{t+a}{(t+a)^2+b^2}$, and that $y(t) = -\frac{b}{(t+a)^2+b^2}$.

b) We show that these solutions must lie on circles in the $x - y$ plane by finding a conservation law for the system in the form of a function $F(x, y)$ which must be constant along the solutions. Hint: reparameterise the solutions, and look for an integrating factor for the resulting differential equation. The effect of reparameterising the trajectories of the system of equations leads to the scalar equation

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2xy}{y^2 - x^2}$$

This equation is not exact, but after studying the form for exactness, we see that we can find an integrating factor $u = u(y)$. Rewriting the equation as a differential form ,and multiplying by the function $u(y)$ we find

$$u(y)\left(y^2 - x^2\right)dy + u(y)(2xy)dx = 0$$

applying the condition for exactness, leads to

$$\begin{aligned}
-2xu(y) &= u'(y)2xy + u(y)2x \\
u'(y)y &= -2u(y) \\
u(y) &= \frac{1}{y^2} \\
\left(1 - \frac{x^2}{y^2}\right) dy &+ \frac{2x}{y} = 0 \\
F(x, y) &= \frac{x^2}{y} + y
\end{aligned}$$

The conservation law for the solutions of the coupled system of differential equations, is $F(x(t), y(t)) = C$. This equation is equivalent to $x^2 + y^2 = Cy$ which is the equation of a circle centered at the point $(0, \frac{C}{2})$ on the y-axis. Thus the solution trajectories must evolve on circles in the x-y plane.
This example is interesting since it constructs a parameterisation of the circles in the plane using rational functions of the parameter t!