Mathematics 231
Introduction to differential equations, Fall, 2009
Solutions Homework 2

1.a) The right hand side of the differential equation is $f(y) = 2ty^2$. This function is continuous throughout the entire $t-y$ plane. The derivative $\frac{\partial f}{\partial y} = 4ty$ which is continuous throughout the entire $t-y$ plane. The largest rectangle where the conclusions of the existence uniqueness theorem apply would then be the entire $t-y$ plane. The existence uniqueness theorem then states, that for any intial condition (a, b) there is a unique solution curve in the $t-y$ plane going through that point.
b) Separating variables, $\frac{dy}{y^2} = 2tdt$ which integrates easily to $-y^{-1} = t^2 + C$. Solving for y after substiting for the intial values to determine C,

$$
\begin{aligned}
-b^{-1} &= a^2 + C \\
C &= -b^{-1} - a^2 \\
y &= \frac{-1}{t^2 - b^{-1} - a^2} \\
y(t, a, b) &= \frac{-b}{bt^2 - 1 - ba^2} \\
bt^2 &= 1 + ba^2, \qquad (1)
\end{aligned}
$$

where the last equation gives the values at the which the denominator of expression for $y(t)$ is zero. The roots of this quadratic expression determine in general three intervals of the time axis. However only one interval contains the value $t = a$ which is essential to determine the interval of existence of the solution $y(t)$.
c) In particular,

$$
\begin{aligned}
b = 0 \quad &\Rightarrow t \in \mathbb{R} \\
b > 0 \quad &\Rightarrow \quad -\sqrt{\frac{1+ba^2}{b}} < a < +\sqrt{\frac{1+ba^2}{b}} \\
b > 0 \quad &\Rightarrow \quad -\sqrt{\frac{1+ba^2}{b}} < t < +\sqrt{\frac{1+ba^2}{b}} \\
b < \frac{-1}{a^2} \quad &\Rightarrow \quad t \in \mathbb{R} \\
\frac{-1}{a^2} < b < 0 \quad &\Rightarrow \quad t \in \mathbb{R} \qquad (2)
\end{aligned}
$$

This calculation above shows how the interval of existence of the solution $y(t, a, b)$ depends on the values of a, b.
d) When $a = 0$ from the formula we developed above, $y = \frac{-b}{bt^2-1}$ which is positive when $b > 0$, and $-\sqrt{\frac{1}{b}} < t < +\sqrt{\frac{1}{b}}$. The denominator has roots at $t = \pm\sqrt{\frac{1}{b}}$. The value $t = 0$ must belong to the interval of existence, so this interval is $-\sqrt{\frac{1}{b}} < t < +\sqrt{\frac{1}{b}}$. When

t approaches the endpoints of its interval of existence, the value of $y \to +\infty$. This is consistent with the existence uniqueness theorem, since the curve leaves its "box" (which is the infinite $t-y$ plane) as the time parameter approaches the endpoints of the interval of existence.

2. a) The function $f(y) = y(y-1)(y-2)$ is a cubic with intercepts on the $y-$axis at $y = 0, 1, 2$. The graph of $f(y)$ versus y therefore looks like an s-shaped curve between its intercepts, with slope 2 at $y = 0$ and $y = 2$, and negative on the intervals $y < 0,\ \ 1 < y < 2$. It is positive when $0 < y < 1, \quad 2 < y$.
b) For the differential equation $y' = f(y)$, the equilibrium points occur at the intercepts of the graph of the function $f(y)$, which in this case are $y = 0, 1, 2$.
c)

$$\begin{aligned} f'(0) &> 0 \text{ unstable} \\ f'(1) &< 0, \text{ stable} \\ f'(2) &> 0, \text{ unstable} \end{aligned} \tag{3}$$

d) phase line diagram

3 a)

$$\begin{aligned} v &= y^{-2} \\ \frac{dv}{dt} &= -2y^{-3}\frac{dy}{dt} \\ &= -2\epsilon y^{-2} + 2\sigma y \\ \\ &= -2\epsilon v + 2\sigma \end{aligned}$$

This is linear with integrating factor $u = e^{2\epsilon t}$. Multiplying with this factor we get

$$\begin{aligned} e^{2\epsilon t}v' + 2\epsilon v e^{2\epsilon t} &= 2\sigma e^{2\epsilon t} \\ v e^{2\epsilon t} &= \frac{2\sigma}{2\epsilon}e^{2\epsilon t} + C \\ v &= \frac{\sigma}{\epsilon} + Ce^{-2\epsilon t} \\ \\ y^2 &= \frac{1}{\frac{\sigma}{\epsilon} + Ce^{-2\epsilon t}} \\ &= \frac{\epsilon}{\sigma + Ce^{-2\epsilon t}} \text{ (relabeling C)} \\ y &= \pm\sqrt{\frac{\epsilon}{\sigma + Ce^{-2\epsilon t}}} \end{aligned}$$

$$
\begin{aligned}
y^2(0) &= \frac{\epsilon}{\sigma + C} \\
C &= \frac{\epsilon}{y_0^2} - \sigma \\
y &= \pm\sqrt{\frac{\epsilon}{\sigma + \left(\frac{\epsilon}{y_0^2} - \sigma\right) e^{-2\epsilon t}}}
\end{aligned}
$$

Where we take postitive square root, if $y_0 > 0$, and take negative square root when $y_0 < 0$.
As $t \to +\infty$, the exponential term dies off, and the asymptotic value of the solution, independent of the intial value $y_0 > 0$ is $+\sqrt{\frac{\epsilon}{\sigma}}$.

b) The equilibrium values of the differential equation are $y_0 = 0, y_0 = \pm\sqrt{\frac{\epsilon}{\sigma}}, \quad \epsilon > 0$. When $\epsilon < 0$, there is only the equilibrium at $y_0 = 0$. The stability of each is determined by the derivative $f'(y_0)$ as follows

$$
f'(0) = \epsilon, \text{ unstable when } \epsilon > 0, \text{ stable when } \epsilon < 0 \tag{5}
$$

$$
\begin{aligned}
f'(+\sqrt{\frac{\epsilon}{\sigma}}) &= \epsilon - 3\sigma\frac{\epsilon}{\sigma} < 0, \text{ stable when } 0 < \epsilon \\
f'(-\sqrt{\frac{\epsilon}{\sigma}}) &= \epsilon - 3\sigma\frac{\epsilon}{\sigma} < 0, \text{ stable when } 0 < \epsilon
\end{aligned} \tag{6}
$$

c) The bifurcation diagram in the $\epsilon - y$ plane describes the equilibrium points, and their stability as the parameter ϵ passes through 0. First the number of equilibrim points changes from one ($\epsilon < 0$) to three ($\epsilon > 0$). The equilibrium point at $y = 0$ changes from stable ($\epsilon < 0$) to unstable ($\epsilon > 0$) and the stable equilibria at $y = \pm\sqrt{\frac{\epsilon}{\sigma}}$ are born at $\epsilon = 0$ and grow as a parabola over the $y-$axis, opening out along the positive ϵ-axis. This type of bifurcation diagram looks like a pitchfork, and is also called a pitchfork bifurcation.