Mathematics 231
Introduction to differential equations, Fall, 2011
Solutions Homework 5

1)

$$y'' + y = \tan(t)$$

We apply the method of variation of parameters, to the homogeneous solution

$$y_h = c_1 \cos(t) + c_2 \sin(t)$$

The equation is already in standard form (coefficient of y'' is 1) so we can write the system of equations for c_1', c_2'

$$\begin{aligned} c_1' \cos(t) + c_2' \sin(t) &= 0 \\ -c_1' \sin t + c_2' \cos(t) &= \tan(t) \end{aligned} \tag{1}$$

One way to solve this system for c_1', c_2' is by Cramer's Rule. The Wronskian of $\cos(t), \sin(t)$ is 1, so

$$\begin{aligned} c_1' &= \begin{vmatrix} 0 & \sin(t) \\ \tan(t) & \cos(t) \end{vmatrix} \\ c_2' &= \begin{vmatrix} \cos(t) & 0 \\ -\sin(t) & \tan(t) \end{vmatrix} \end{aligned}$$

which gives

$$\begin{aligned} c_1' &= -\frac{\sin^2(t)}{\cos(t)} \\ &= -\frac{1-\cos^2(t)}{\cos(t)} \\ &= -\sec(t) + \cos(t) \\ c_2' &= \sin(t) \end{aligned} \tag{2}$$

This gives $c_1 = -\ln|\sec(t) + \tan(t)| + \sin(t)$ (standard form from table of integrals) and $c_2 = -\cos(t)$.

$$y_p = -\cos(t)\ln|\sec(t)+\tan(t)| + \cos(t)\sin(t) - \cos(t)\sin(t) = -\cos(t)\ln|\sec(t)+\tan(t)|$$

The general solution to the nonhomogeneous problem then becomes

$$y = c_1\cos(t) + c_2\sin(t) - \cos(t)\ln|\sec(t)+\tan(t)|, \quad -frac\pi 2 < t < \frac{\pi}{2}$$

2) We will choose a coordinate system so that the downward direction is positive and the upward direction is negative. The displacement of the mass from its equilibrium position is $y(t)$.

$$\text{(DE) } my'' + by' + ky = 0,$$

is the governing equation for this mechanical system. We need to determine the parameters in accord with the information which is given to us in the problem. The parameters are determined from the conditions

$$m = 2 \text{ kg}, \quad b = \frac{3}{5} \text{ N-sec/m}, \quad \frac{k}{10} = 3$$

which gives the characteristic equation $2r^2 + \frac{3}{5}r + 30 = 0$. The roots of this quadratic are

$$r = \frac{-3}{20} \pm i\sqrt{\frac{5991}{400}} = -.15 \pm 3.870i$$

The general solution is

$$y = c_1 e^{-.15t}\cos(3.870t) + c_2 e^{-.15t}\sin(3.870t)$$

The intitial conditions are

$$y(0) = .05 \text{ m}, \quad y'(0) = -0.1 \text{ m}$$

The parameters c_1, c_2 can then be determined from

$$\begin{aligned} c_1 &= 0.05 \\ -0.15c_1 + 3.870c_2 &= -0.1 \\ c_2 &= -0.0239 \end{aligned} \tag{3}$$

The solution for the initial value problem is

$$\begin{aligned} y(t) &= 0.05e^{-.15t}\cos(3.870t) - 0.02390e^{-.15t}\sin(3.870t) \\ &= e^{-.15t}\left[(.05)^2 + (0.0239)^2\right]^{\frac{1}{2}}\cos(3.870t \; - \phi) \\ &= 0.05541e^{-.15t}\cos(3.870t + 0.4458) \\ \phi &= \arctan\left(\frac{-0.02390}{0.05}\right) \\ &= -0.4458 \end{aligned} \tag{4}$$

The quasiperiod is $\frac{2\pi}{3.870}$

3) For the differential equation, we let $y_p(t)$ denote the particular solution

$$\text{(DE)} \; t^2y'' + 7ty' + 5y = t, \quad t > 0$$

To get the homogeneous solution we substitute t^α into (DE) to obtain $\alpha(\alpha-1)+7\alpha+5 = (\alpha+5)(\alpha+1) = 0$. The correpsonding homogeneous solution is $y_h = c_1t^{-5} + c_2t^{-1}$

To use the method of variation of parameters, it is suggested to write (DE) in standard form, which is

$$y'' + \frac{7}{t}y' + \frac{5}{t} = \frac{1}{t}.$$

The linear equations for the derivatives of $c_1(t), c_2(t)$ then become

$$\begin{aligned} c_1' t^{-5} + c_2' t^{-1} &= 0 \\ -5t^{-6}c_1' - t^{-2}c_2' &= t^{-1} \end{aligned} \tag{5}$$

The matrix of coefficients is nonsingular and has determinant equal to

$$W\left[t^{-5}, t^{-1}\right] = \begin{vmatrix} t^{-5} & t^{-1} \\ -5t^{-6} & -t^{-2} \end{vmatrix} = 4t^{-7}$$

The solution to hte linear system is done with Cramer's rule (since we already have the determinant)

$$\begin{aligned} 4t^{-7}c_1' &= \begin{vmatrix} 0 & t^{-1} \\ t^{-1} & -t^{-2} \end{vmatrix} \\ &= -t^{-2} \\ c_1' &= \frac{-1}{4}t^5 \\ c_1 &= \frac{-1}{24}t^6 \\ 4t^{-7}c_2' &= \begin{vmatrix} t^{-5} & 0 \\ -5t^{-6} & t^{-1} \end{vmatrix} \\ &= t^{-6} \\ c_2' &= \frac{1}{4}t \\ c_2 &= \frac{1}{8}t^2 \end{aligned} \tag{6}$$

This gives a particular solution

$$y_p = c_1 t^{-5} + c_2 t^{-1} = \frac{1}{12}t$$

and the corresponding general solution

$$y = y_h + y_p = c_1 t^{-5} + c_2 t^{-1} + \frac{1}{12}t$$

4) We will compute the solution of

$$x' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} x$$

First we calculate the characteristic polynomial of the matrix of coefficients,

$$\det\,(A - \lambda I) = \left(\frac{5}{4} - \lambda\right)^2 - \frac{9}{16}$$

The roots of the characteristic polynomial are

$$\lambda = \frac{5}{4} \pm \frac{3}{4} = \frac{1}{2},\quad 2$$

The corresponding eigenvectors are computed as usual:

$$\left(A - \frac{1}{2}I\right)\xi = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvector is $\xi_1 = (1, -1)^T$.

$$(A - 2I)\,\xi = \begin{pmatrix} \frac{-3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{-3}{4} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvector is $\xi_2 = (1, 1)^T$.
The general solution is

$$x(t) = c_1 e^{\frac{t}{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

All solutions except $x = 0$ tend to ∞ as $t \to +\infty$ and all solutions tend to 0 as $t \to -\infty$. The sketch of the phase plane trajectories follows.