Math 237, Introduction to Differential Equations, Fall 2011

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Homework 3, Solutions

1 a) Find the general solution on the inteval $-\infty < t < +\infty$

$$y'' - 2y' - 3y = 0$$

The characteristic polynomial is $r^2 - 2r - 3 = (r-3)(r+1)$. The roots are $r_1 = -1, r_2 = 3$. Corresponding exponential solutions are $y_1 = e^{-t}$ and $y_2 = e^{3t}$. The Wronskian determinant of these solutions at $t = 0$ is

$$W[e^{-t}, e^{3t}](0) = \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 4$$

The solutions form a fundamental set , and the general solution is $y(t) = c_1 e^{-t} + c_2 e^{3t}$.

b) Find the general solution

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$$

The characteristic polynomial is $2r^2 + 3r + 1 = (2r+1)(r+1)$. Roots are $r_1 = -1, r_2 = -\frac{1}{2}$. The corresponding exponential solutions and their Wronskian at $t = 0$

$$W[e^{-x}, e^{-\frac{x}{2}}](0) = \begin{vmatrix} 1 & 1 \\ -1 & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

The general solution is a linear combination $y(x) = c_1 e^{-x} + c_2 e^{3x}$. **c)** Find two linearly independent solutions (show that the Wronskian is not 0 for all t)

$$(D^2 + 2D + 1)[y] = 0, \quad D = \frac{d}{dt}$$

The characteristic polynomial is $r^2 + 2r + 1 = (r + 1)^2$. Using the exponential shift we observe that

$$(D + 1)^2[ve^{-t}] = e^{-t} D^2[v] = 0$$

To get solutions we can take $D^2[v] = 0$ or $v = c_1 + c_2 t$. This gives two solutions $y_1 = e^{-t}$ and $y_2 = te^{-t}$. The Wronskian determinant of these two solutions is

$$W[e^{-t}, te^{-t}] = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & -te^{-t} + e^{-t} \end{vmatrix} = -te^{-2t} + e^{-2t} + te^{-2t} = e^{-2t}$$

The two solutions are linearly independent on $(-\infty, +\infty)$.

2. Find the solution of the intial value problem, and indicate on what interval it is valid.

$$y'' + 2y' - 5y = 0, \quad y(0) = 1, \quad y'(0) = -1$$

The characteristic polynomial is $r^2 + 2r - 5 = (r+1)^2 - 6$ the roots are $r_{2,1} = -1 \pm \sqrt{6}$. The Wronskian is

$$W[e^{r_1}, e^{r_2}](0) = \begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = (r_2 - r_1) = 2\sqrt{6}$$

We now consider a system of linear equations in coefficients c_1, c_2 to match the IVP

$$c_1 + c_2 = 1$$

$$r_1 c_1 + r_2 c_2 = -1$$

the solution to this system of equations is by Cramers Rule

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ -1 & r_2 \end{vmatrix}}{2\sqrt{6}}, \quad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ r_1 & -1 \end{vmatrix}}{2\sqrt{6}}$$

This gives the solution $c_1 = \frac{r_2+1}{2\sqrt{6}} = \frac{1}{2}$ and $c_2 = \frac{-r_1-1}{2\sqrt{6}} = \frac{1}{2}$ and the corresponding solution which is valid on $(-\infty, +\infty)$,

$$y(t) = \frac{1}{2}e^{-t-\sqrt{6}t} + \frac{1}{2}e^{-t+\sqrt{6}t}$$

3. Using the exponential shift find three linearly independent solutions of

$$(D^3 - 2D^2 + D)[y] = 0, \quad D = \frac{d}{dx}$$

The Differential operator can be factored

$$P(D) = D(D-1)^2$$

Two solutions to the homogeneous equation are therefore $y_1(x) = 1, y_2(x) = e^x$. To get a third we use the exponential shift applied to a product $v(x)e^x$

$$D(D-1)^2[ve^x) = e^x(D+1)D^2[v] = 0$$

This allows us to get the solution $v = x$ or $v = e^{-x}$. The resulting solutions are $y_3 = xe^x$ or $y_3 = e^x E^{-x}$ but the second of these solutions has already been found $y_1 = 1$. The Wronskian determinant of these three solutions

$$W[1, e^x, xe^x](0) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} = 1$$

These solutions are thereby linearly independent on $(-\infty, +\infty)$. **4.** Consider the third order linear homogeneous differential equation

$$(D^3 + 4D^2 + D - 6)[y] = 0.$$

a) Show that $y = e^t$ is a solution to this equation. Use this to factor the characteristic equation and find two additional exponential solutions.

b) Use the solutions you have found in part a) to find a unique solution for the initial value problem

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 1$$

a $P(D)[e^t] = (D^3 + 4D^2 + D - 6)[e^t] = (6 - 6)e^t = 0$. The differential operator can therefore be factored

$$P(D) = (D - 1)(D^2 + 5D + 6) = (D - 1)(D + 2)(D + 3)$$

The corresponding exponential solutions are $y_3 = e^t, y_2 = e^{-2t}, y_1 = e^{-3t)}$. The

Wronskian is

$$W[e^{-3t}, e^{-2t}, e^{t}](0) = \begin{vmatrix} 1 & 1 & 1 \\ -3 & -2 & 1 \\ 9 & 4 & 1 \end{vmatrix} = (1+3)(1+2)(-2+3) = 12$$

These solutions are thereby linearly independent on $(-\infty, +\infty)$.

b) We set up a system of three linear equations in unknowns c_1, c_2, c_3 to match the IVP

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ -3c_1 - 2c_2 + c_3 &= 0 \\ 9c_1 + 4c_2 + c_3 &= 1 \end{aligned}$$

The solution by Cramers Rule (using the denominator determinant as in calculation of the Wronskian)

$$c_1 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 1 & 4 & 1 \end{vmatrix}}{12}, \quad c_2 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -3 & 0 & 1 \\ 9 & 1 & 1 \end{vmatrix}}{12}, \quad c_3 = \frac{\begin{vmatrix} 1 & 1 & 1 \\ -3 & -2 & 0 \\ 9 & 4 & 1 \end{vmatrix}}{12}$$

This gives the solution $c_1 = -\frac{3}{12}, c_2 = \frac{8}{12}, c_3 = \frac{7}{12}$