Math 237, Introduction to Differential Equations, Fall 2011

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Homework 4, Solutions

1. a) Find the general solution to the homogeneous equation

$$\left(D^3 + D^2 + 3D - 5\right)[y] = 0$$

using the fact that the differential operator has a factor (D-1).

The differential operator factors $P(D) = (D-1)(D^2+2D+5) = (D-1)((D+1)^2+4)$

The corresponding solutions to the homoegeneous equation are $y_1(t) = e^t, y_2(t) = e^{-t}\cos(2t), y_3(t) = e^{-t}\sin(2t)$. The last two are the real and imaginary parts of the complex exponentials $e^{(-1\pm i2)t}$. These three (real valued) solutions are linearly independent since the Wronskian determinant is not 0 at $t = 0$

$$W\left(y_1, y_2, y_3\right)(0) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & -3 & -4 \end{vmatrix} = 16$$

Thus the general solution consists of a linear combination of the three fundamental solutions

$$y(t) = c_1 e^t + c_2 e^{-t}\cos(2t) + c_3 e^{-t}\sin(2t)$$

b) By factoring the differential operator and using the exponential shift, find four real valued linearly independent solutions to the homogenous equation

$$\left(D^4 + 2D^2 + 1\right)[y] = 0, \quad D = \frac{d}{dx}$$

The differential operator factors $P(D) = \left(D^2 + 1\right)^2$. Thus there are solutions $y_1(x) = \cos(x), y_2(x) = \sin(x)$,which are the real and imaginary parts of the complex exponential e^{ix}, and two further solutions $y_3(x) = x\cos(x), y_4(x) = x\sin(x)$. The last two solutions can be seen to be solutions, either by direct calculation with the differential operator $P(D)$, or by applying the exponential shift to the complex exponential functions

$$P(D)[xe^{ix}] = e^{ix}P(D+i)[x] = e^{ix}(D+2i)^2D^2[x] = 0$$

Taking real and imaginary parts of the complex exponential function, we see that $y_3(x) = x\cos(x)$ and $y_4(x) = x\sin(x)$ are also solutions of the homogeneous equation. To show linear independence on the real line we need only compute the Wronskian at one point, and show it is not 0 at that point

$$W[y_1, y_2, y_3, y_4](0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & -3 & 3 \end{vmatrix} = 2$$

2. Using the methods of undetermined coefficients, find the solution to the nonhomogeneous initial value problem

$$\left(D^2 + 3D + 2\right)[y] = \sin(x), \quad y(0) = 0, \quad y'(0) = 0, \quad D = \frac{d}{dx}$$

First we complete the square in the characteristic equation $P(r) = r^2 + 3r + 2 = (r + \frac{3}{2})^2 - \frac{1}{4}$. The roots of this equation are $r_1 = -2, r_2 = -1$. The homogeneous solution is $y_h(x) = c_1 e^{-2x} + c_2 e^{-x}$. To constuct a particular solution we can use the annihilator method. (It is not necessary to include these details, but i put them to remind students of our classroom discussions). We look for an annihilator of the right hand side, and see that $(D^2 + 1)[\sin(x)] = 0$. Thus the annihilator is $D^2 + 1$ and we apply this operator to each side of the differential equation. We see that the particular solution we are looking for must satisfy

$$(D^2 + 1)\left((D + \frac{3}{2})^2 - \frac{1}{4}\right)[y_p(x)] = 0$$

Ignoring the solutions of this homogeneous equation which correspond to the homogeneous solutions $Y_h(x)$ we have already found, it must be that

$$y_p(x) = A\cos(x) + B\sin(x).$$

Substituting this expression into the left hand side of the nonhomogeneous equation above, we find after simplification

$$(-A + 3B + 2)\cos(x) + (-B - 3A + 2)\sin(x) = \sin(x),$$

$$-A + 3B = -2, \quad -B - 3A = -1, \quad -10A = -5, \quad -B = \frac{1}{2}$$

The general solution is now $y_h + y_p$

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{(}{1} 2\cos(x) - \frac{1}{2}\sin(x)$$

and applying the initial conditions we find that

$$c_1 + c_2 = -\frac{1}{2}, \quad -2c_1 - c_2 = \frac{1}{2}$$

We find the unique solution to be $c_1 = 0, c_2 = -\frac{1}{2}$.

$$y(x) = -\frac{1}{2}e^{-x} + \frac{(}{1} 2\cos(x) - \frac{1}{2}\sin(x)$$

3. Using your answer from question 2 and the principle of superposition for nonhomogenous equations, find the general solution to the equation

$$\left(D^2 + 3D + 2\right)[y] = \sin(x) + 10e^{3x}$$

We apply the principle of superpostion and determine that the particular solution must be of the form

$$y_p(x) = Ae^{3x} + \frac{(}{1} 2\cos(x) - \frac{1}{2}\sin(x)$$

we apply the differential operator to determine that the coefficient A must satisfy $9A + 9A + 2A = 10$, or $A = \frac{1}{2}$. The general solution to the nonhomogeneous equation is therefore

$$y(x) = c_1 e^{-2x} + c_2 e^{-x} + \frac{(}{1} 2\cos(x) - \frac{1}{2}\sin(x) + \frac{1}{2}e^{3x}$$

4. Using the exponential shift again find the solution to the initial value problem

$$\left(D^2 - 1\right)[y] = 3e^t, \quad y(0) = 0, y'(0) = -1, \quad D = \frac{d}{dt}$$

The homogeneous solution is $y_h(t) = c_1 e^{-t} + c_2 e^t$. Since the annihilator of $3e^t$ is $(D-1)$, it follows that the annihilator of $y_p(t)$ is $(D-1)\left(D^2-1\right) = (D-1)^2(D+1)$. We see that the right hand side is a solution to the homogeneous equation. We may use the exponential shift theorem (but we dont necessarily have to apply this method) to see that $y_p(t) = (B + At)e^t$. Ignoring the solutions of the homogeneous equation we determine that the simplest form of the particular solution is $y_p(t) = Ate^t$. This is substituted into the differential equation to find that $2Ae^t + Ate^t - Ate^t = 3e^t$. From this we deduce that $A = \frac{3}{2}$ so

$$y(t) = c_1 e^{-t} + c_2 e^t + \frac{3}{2} te^t$$

Now matching the intial conditions we find that

$$c_1 + c_2 = 0, \quad -c_1 + c_2 = -\frac{3}{2}, \quad c_1 = \frac{3}{4}, c_2 = -\frac{3}{4}$$