Mathematics 237
Introduction to differential equations, Fall, 2011
Solutions Homework 5

1 We will choose a coordinate system so that the downward direction is positive and the upward direction is negative. The displacement of the mass from its equilibrium position is $y(t)$.

$$\text{(DE) } my'' + by' + ky = 0,$$

is the governing equation for this mechanical system. We need to determine the parameters in accord with the information which is given to us in the problem. The parameters m, b, k are determined from the information given in the problem

$$\begin{aligned} m &= 0.5 \text{ slug} \\ b &= 4 \text{ lbs-sec/foot} \\ 2k &= \text{ weight (lbs)} = \text{ mass times acceleration} \\ &= \frac{32}{2} = 16 \\ k &= 8 \text{ lbs/foot} \end{aligned} \quad (1)$$

The equation of motion becomes

$$y'' + 8y' + 16y = 0$$

The roots of the characteristic equation are $r = -4$ repeated with multiplicity 2. The general solution is

$$y = (c_1 + tc_2)e^{-4t}$$

The initial conditions are

$$y(0) = -0.5, \quad y'(0) = 0$$

This gives the values of

$$c_1 = -0.5, \quad c_2 = 0.5$$

and the solution of the initial value problem is

$$y(t) = (-0.5 + 0.5t)e^{-4t}$$

The solution has exactly one zero when the mass crosses the equilibrium point at $t = 1$ second. There are no oscillations.

2)

$$\text{(DE) } y''' - 2y'' + y' = t^3 + 2e^t$$

The characteristic equation of the homogeneous equation can be factored

$$r^3 - 2r^2 + r = r(r^2 - 2r + 1) = r(r-1)^2 = 0$$

The corresponding homogeneous solution is $y_h = c_1 + (c_2 + tc_3)e^t$. We need this even if we are only looking for the particular solution. The particular solution will be of the form $y_p = y_1 + y_2$ where

$$\begin{aligned} y_1''' - 2y_1'' + y_1' &= t^3 \\ y_2''' - 2y_2'' + y_2' &= 2e^t \\ D^4(D^3 - 2D^2 + D)y_1 &= 0 \\ D^5(D-1)^2 y_1 &= 0 \\ (D-1)(D^3 - 2D^2 + D)y_2 &= 0 \\ D(D-1)^3 y_2 &= 0 \end{aligned} \qquad (2)$$

Looking at these two equations for y_1, y_2 , we write the general form of the solution, and discard any parts which are homogeneous solutions. We find that

$$\begin{aligned} y_1 &= At + Bt^2 + Ct^3 + Dt^4 \\ y_2 &= Et^2 e^t \\ y_p &= At + Bt^2 + Ct^3 + Dt^4 + Et^2 e^t \end{aligned} \qquad (3)$$

3) For the differential equation, we let $y_p(t)$ denote the particular solution

$$\text{(DE)}\ t^2 y'' + 7ty' + 5y = t, \quad t > 0$$

To get the homogeneous solution we substitute t^α into (DE) to obtain $\alpha(\alpha-1) + 7\alpha + 5 = (\alpha+5)(\alpha+1) = 0$. The correpsonding homogeneous solution is $y_h = c_1 t^{-5} + c_2 t^{-1}$
To use the method of variation of parameters, it is suggested to write (DE) in standard form, which is

$$y'' + \frac{7}{t}y' + \frac{5}{t} = \frac{1}{t}.$$

The linear equations for the derivatives of $c_1(t), c_2(t)$ then become

$$\begin{aligned} c_1' t^{-5} + c_2' t^{-1} &= 0 \\ -5t^{-6} c_1' - t^{-2} c_2' &= t^{-1} \end{aligned} \qquad (4)$$

The matrix of coefficients is nonsingular and has determinant equal to

$$W\left[t^{-5}, t^{-1}\right] = \begin{vmatrix} t^{-5} & t^{-1} \\ -5t^{-6} & -t^{-2} \end{vmatrix} = 4t^{-7}$$

The solution to hte linear system is done with Cramer's rule (since we already have the determinant)

$$4t^{-7} c_1' = \begin{vmatrix} 0 & t^{-1} \\ t^{-1} & -t^{-2} \end{vmatrix}$$

$$
\begin{aligned}
&= -t^{-2} \\
c_1' &= \frac{-1}{4}t^5 \\
c_1 &= \frac{-1}{24}t^6 \\
4t^{-7}c_2' &= \begin{vmatrix} t^{-5} & 0 \\ -5t^{-6} & t^{-1} \end{vmatrix} \\
&= t^{-6} \\
c_2' &= \frac{1}{4}t \\
c_2 &= \frac{1}{8}t^2
\end{aligned} \tag{5}
$$

This gives a particular solution

$$y_p = c_1 t^{-5} + c_2 t^{-1} = \frac{1}{12}t$$

and the corresponding general solution

$$y = y_h + y_p = c_1 t^{-5} + c_2 t^{-1} + \frac{1}{12}t$$

4) We will compute the solution of

$$\text{DE } x' = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{5}{4} \end{pmatrix} x$$

First we calculate the characteristic polynomial of the matrix of coefficients,

$$\det\,(A - \lambda I) = \left(\frac{5}{4} - \lambda\right)^2 - \frac{9}{16}$$

The roots of the characteristic polynomial are

$$\lambda = \frac{5}{4} \pm \frac{3}{4} = \frac{1}{2}, \quad 2$$

The corresponding eigenvectors are computed as usual:

$$\left(A - \frac{1}{2}I\right)\xi = \begin{pmatrix} \frac{3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvector is $\xi_1 = (1, -1)^T$.

$$(A - 2I)\,\xi = \begin{pmatrix} \frac{-3}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{-3}{4} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvector is $\xi_2 = (1, 1)^T$.
The general solution is

$$x(t) = c_1 e^{\frac{t}{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

All solutions except $x = 0$ tend to ∞ as $t \to +\infty$ and all solutions tend to 0 as $t \to -\infty$. The sketch of the solutions should show a dominant direction as $t \to -\infty$. This is the direction $\xi_1 = (1, -1)^T$ since the corresponding eigenvalue is smallest in absolute value.