# Solutions to Assignment 7

**Problem 1**

Show that the Laplace transform of $\cos(\alpha t)$ satisfies

$$\mathcal{L}\{\cos \alpha t\} = \frac{s}{s^2+\alpha^2}$$

**Solution:**

$$\begin{aligned}
\mathcal{L}(\cos \alpha t) \quad &= \lim_{A\to\infty} \int_0^A e^{-st}\cos(\alpha t)dt \\
&= \lim_{A\to\infty} e^{-st}\frac{\sin(\alpha t)}{\alpha}|_0^A - \int_0^A \frac{-s}{\alpha}e^{-st}\sin(\alpha t)dt \\
&= \lim_{A\to\infty} e^{-st}\frac{\sin(\alpha t)}{\alpha}|_0^A + \left\{e^{-st}\cos(\alpha t)\frac{s}{\alpha^2}|_0^A - \int_0^A \frac{s}{\alpha^2}e^{-st}\cos(\alpha t)dt\right\} \\
&= \lim_{A\to\infty} e^{-st}\frac{\sin(\alpha t)}{\alpha}|_0^A + \frac{s}{\alpha^2}\left\{e^{-st}\cos(\alpha t)|_0^A - s\int_0^A e^{-st}\cos(\alpha t)dt\right\} \\
&= \lim_{A\to\infty} e^{-st}\frac{\sin(\alpha t)}{\alpha}|_0^A + \lim_{A\to\infty}\frac{s}{\alpha^2}\left\{e^{-st}\cos(\alpha t)|_0^A - \int_0^A se^{-st}\cos(\alpha t)dt\right\} \\
&= \frac{s}{\alpha^2} - \frac{s^2}{\alpha^2}\mathcal{L}(\cos \alpha t)\}
\end{aligned} \tag{1}$$

Hence,

$$\mathcal{L}(\cos \alpha t)(1+\frac{s^2}{\alpha^2}) = \frac{s}{\alpha^2},$$

and the result follows by dividing both sides by $(1+\frac{s^2}{\alpha^2})$.

**Problem 2**

Use the Laplace transform method to obtain the solution to the following:

$$y'' - 2y' + 4y = 0, \qquad y(0) = 2, y'(0) = 1$$

**Solution:**

Taking the Laplace transform of $y$:

$$\mathcal{L}\{y'\}(s) = s\mathcal{L}\{y\} - y(0)$$
$$\mathcal{L}\{y''\}(s) = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

leads to:

$$\mathcal{L}\{y\}(s^2 - 2s + 4) = -2y(0) + y'(0)$$

Hence,

$$\mathcal{L}\{y\}(s) = 2\frac{4(s-2)}{s^2 - 2s + 4} = \frac{s-1}{(s-1)^2 + 3} - \frac{1}{(s-1)^2 + 3}$$

By observing that $s - 1$ term gives rise to a sine and a cosine, and a shift in the $s-$domain is equivalent to multiplication by an exponential in the original function, we obtain,

$$y(t) = 2e^t \cos(\sqrt{3}t) - \frac{1}{\sqrt{3}}e^t \sin(\sqrt{3}t)$$

**Problem 3**

Suppose that

$$g(t) = \int_0^t f(\tau)d\tau.$$

If $G(s)$ and $F(s)$ are the Laplace transforms of $g(t)$ and $f(t)$, show that

$$G(s) = F(s)/s$$

**Solution:**

It could be noticed that

$$g(t) = \int_0^t f(s)ds$$

implies that $g'(t) = f(t)$. Hence, $F(s) = sG(s)$.

**Problem 4**

Suppose that

$$F(s) = \mathcal{L}\{f(t)\},$$

exists for $s > 0$.

a) Show that if $c > 0$ then, for $s > 0$,

$$\mathcal{L}\{f(ct)\} = \frac{1}{c}F(\frac{s}{c})$$

b) Show that if $k > 0$ then, for $s > 0$,

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k}f(\frac{t}{k})$$

b) Show that if $a, b > 0$ are constants, then, for $s > 0$,

$$\mathcal{L}^{-1}\{F(as+b)\} = \frac{1}{a}e^{-bt/a}f(\frac{t}{a})$$

**Solution:**

a) In the following, let $u = ct$:

$$\begin{aligned}
\mathcal{L}(f(ct))(s) \quad &= \lim_{A\to\infty}\int_0^A e^{-st}f(ct)dt \\
&= \lim_{A\to\infty}\int_0^{A/c} e^{-s(u/c)f(u)\frac{du}{c}} \\
&= \lim_{A\to\infty}\int_0^{A/c} e^{-s/cu}f(u)\frac{du}{c} \\
&= \frac{1}{c}F(\frac{s}{c})
\end{aligned} \tag{2}$$

b) This part follows from the result above, that is the Laplace transform of $\frac{1}{k}f(\frac{t}{k})$ equals $\frac{k}{k}F(ks) = F(ks)$

c) Consider the Laplace transform

$$\mathcal{L}(\frac{1}{a}e^{-bt/a}f(t/a))(s)$$

We know the following:

$$\begin{aligned}
\mathcal{L}\{(\frac{1}{a}e^{-bt/a}f(t/a))\}(s) \quad &= \quad \lim_{A\to\infty}\int_0^{aA} e^{-st}(\frac{1}{a}e^{-bt/a}f(t/a)) \\
&= \quad \frac{1}{a}\lim_{A\to\infty}\int_0^{aA} e^{-(s+b/a)t}f(t/a)
\end{aligned}$$

Now, defining a variable $u = t/a$, and applying the steps in part a, it follows that

$$\begin{aligned}
\mathcal{L}\{(\frac{1}{a}e^{-bt/a}f(t/a))\}(s) \quad &= \frac{1}{a}\lim_{A\to\infty}\int_0^{aA} e^{(as+b)u}f(u)adu \\
&= F(as+b)
\end{aligned} \tag{3}$$

**Problem 5**

Compute the inverse Laplace transform of

$$\frac{s^2+9s+2}{(s-1)^2(s+3)},$$

where the inverse is a continuous function.

Hint: Use partial fraction expansion and the properties of the derivative of a Laplace transform.

**Solution:**

We express

$$\frac{s^2+9s+2}{(s-1)^2(s+3)} = \frac{a}{(s-1)^2} + \frac{b}{s-1} + \frac{c}{s+3},$$

and find $a = 2, b = 3, c = -2$

The inverse transform leads to:

$$2e^t + 3te^t - 2e^{-3t}$$

**Problem 6**

The transfer function, $H(s)$, of a linear system is defined as the ratio of the Laplace transform of the output $y(t)$ to the Laplace transform of the input function $g(t)$, under the assumption that all the initial conditions are set to zero. If the linear system is governed by a differential equation:

$$ay'' + by' + cy = g(t)$$

Verify that the transfer function is given by:

$$H(s) = \frac{1}{as^2 + bs + c}$$

Now suppose the Laplace transform of $g(t)$ satisfies $G(s) = 1$. In this case, you can see that $y(t)$ is the inverse Laplace of the transfer function, which we denote by $h(t)$. But, $G(s) = 1$ is the Laplace transform of the impulse. This is why $h(t)$ is called the impulse response of the system.

**Solution:**

Taking the Laplace transform of both sides, and observing that the initial conditions are set to zero, we obtain

$$\mathcal{L}(ay'' + by' + cy) = \mathcal{L}(g)$$

and

$$\mathcal{L}(y) = \frac{\mathcal{L}(g)}{(as^2 + bs + c)}.$$

Hence, if $G(s) = 1$, we obtain:

$$H(s) = \frac{1}{(as^2 + bs + c)}$$

$H(s)$ is the transfer function; it multiplies the Laplace transform of the input to generate the Laplace transform of the output at a given value of $s$.

* * * Have a good, well-deserved winter break * * *