Department of Mathematics

Queen's University

MTHE 237

Midterm Examination Fall 2011

Do any four of the following five questions

1. [10 marks] The following equation is not exact, but can be made exact with an integrating factor of the form $u = u(t)$. Find this integrating factor, then find the general solution of the corresponding differential equation

$$\left(4t + 2e^t\right) ydt + \left(3t^2 + 3e^t\right) dy = 0.$$

The following equation is not exact, but can be made exact with an integrating factor of the form $u = u(x)$. **a)** Find this integrating factor. **b)** Find the general solution of the differential equation

$$\mathbf{(4x + 2e^x)\, ydx + \left(3x^2 + 3e^x\right) dy = 0}.$$

a) Multiply the entire differential equation by $U(x)$ and apply the condition of exactness

$$U(x)(4x + 2e^x) = U'(x)\left(3x^2 + 3e^x\right) + U(x)\left(6x + 3e^x\right)$$

simplifying and solving for $U'(x)$ we get

$$U'(x) = U(x)\frac{-2x - e^x}{3x^2 + 3e^x}$$

Separating variables, and integrating we find

$$\ln(U(x)) = \int \frac{-2x - e^x}{3x^2 + 3e^x} dx = -\frac{1}{3}\ln(3x^2 + 3e^x)$$

From which it follows that

$$U(x) = (3x^2 + 3e^x)^{\frac{-1}{3}}$$

b Multiply the differential equation by this factor, we can integrate the term $N(x, y)$ to get

$$F(x, y) = \int (3x^2 + 3e^x)^{\frac{-1}{3}} \left(3x^2 + 3e^x\right) dy = (3x^2 + 3e^x)^{\frac{2}{3}} y + h(x)$$

We can check that $h'(x) = 0$ and we get the general solution

$$(3x^2 + 3e^x)^{\frac{2}{3}} y = C$$

2. [10 marks] Find the largest rectangular region in the (t, y) plane which contains the point $t = 0, y = 1$ and on which the hypotheses of the existence-uniqeness theorem holds for the solution of the initial value problem

$$\frac{dy}{dt} = y^2 t, \quad y(0) = 1.$$

Find the solution and determine on what interval of time it is defined and exists.

The right hand side of the differential equation together with its partial derivative is

$$f(t, y) = y^2 t, \quad f_y(t, y) = 2yt$$

which are continuous functions on the entire t-y plane. By the statement of the existence uniqueness theorem, the largest rectangle on which the conclusions of this theorem are valid is the entire t-y plane.

The differential equation is separable and therefore exact. We separate the variables, and integrate

$$\frac{1}{y^2} dy = t dt, \quad \frac{-1}{y} = \frac{1}{2} t^2 + C$$

The intial condition can be applied to evaluate the constant of integration C; rearranging the solution

$$-1 = C, \quad y(t) = \frac{-1}{\frac{1}{2}t^2 - 1}$$

The solution is valid on the interval containing $t = 0$ for which $\frac{1}{2}t^2 - 1 \neq 0$ This coincides with the interal

$$-\sqrt{2} < t < +\sqrt{2}$$

3. [10 marks] Suppose that we set up an simple damped spring-mass system with a 4kg mass attached to a spring and sretched by 1m from its equilibrium configuration and then released with zero velocity. If the spring constant is $k = 1\frac{\text{N}}{\text{m}}$ and the damping coefficient is $b = 4\frac{\text{N}}{\text{m/s}}$

a) Find the governing differential equation and the intitial conditions for this mechanical system in terms of the displacement from equilibrium.

b) Find the displacement from equilibrium for any time t, and the time T at which the mass will return to its equilibrium configuration.

a The differential equation in terms of the diplacement from equilibrium $y(t)$, is

$$\left(mD^2 + bD + k\right)[y] = 0, \quad \left(4D^2 + 4D + 1\right)[y] = 0$$

The initial conditions are

$$y(0) = 1, \quad y'(0) = 0$$

b The characteristic equation is $4r^2 + 4r + 1 = 4(r^2 + r + \frac{1}{4}) = 4(r + \frac{1}{2})^2$. The characteristic root $r = -\frac{1}{2}$ is repeated, and the two linearly independent solutions of a fundamental set are

$$y_1 = e^{-\frac{1}{2}t}, \quad y_2 = te^{-\frac{1}{2}t}$$

The general solution is

$$y(t) = (c_1 + tc_2)e^{-\frac{1}{2}t}$$

We apply the initial conditions and find that

$$c_1 = 1, \quad -\frac{1}{2}c_1 + c_2 = 0, \quad c_2 = \frac{1}{2}$$

The return time T to equilibrium can be calculated by setting

$$1 + \frac{1}{2}T = 0, \quad T = -2$$

4. [10marks] Show that $y = e^{-t}$ is a solution of the homogeneous equation

$$y''' + y'' - 4y' - 4y = 0$$

a) Find three linearly independent solutions of the differential equation, and give careful reasons as to why they are linearly independent and on what interval they have this property.

b) Give the system (without solving it) of linear equations which may be used to solve the initial value problem $y(1) = 2, y'(1) = -1, y''(1) = 4$

The polynomial differential operator factors like $P(D) = (D+1)(D^2-4) = (D+1)(D-2)(D+2)$. The corresponding exponential solutions are $y_1 = e^{-t}, y_2 = e^{-2t}, y_3 = e^{2t}$. To show that they are linearly independent on the entire real line, we need only compute the Wronskian determinant at one point to see if it 0 or not 0.

$$W[y_1, y_2, y_3](0) = \begin{vmatrix} 1 & 1 & 1 \\ -1 & -2 & 2 \\ 1 & 4 & 4 \end{vmatrix} = -12$$

The solutions y_1, y_2, y_3 form a fundamental set of solutions on the entire real line (by Abel's theorem).

5. [10 marks] A tank contains 100 liters of water in which 20 gms of salt are dissolved. A brine solution containing 2 gms per liter of salt is poured into the tank at the rate of 4 liters per minute, and the well stirred mixture flows out at the rate of 3 liters per minute from a spigot at the bottom of the tank.

a) Write down a differential equation for the amount $S(t)$ of salt (gms) in solution at any time.

b) Solving this differential equation, find the amount of salt $S(t)$(gms) present in the tank at any time after the process begins. What is the long time limit as $t \to +\infty$ of the concentration of salt in solution?

a Let $S(t)$ denote the amount of salt in solution measured in grams. The volume of brine is increasing at the rate of 1 liter per minute (rate in - rate out $= 1$). Thus the volume of brine is $V(t) = 100 + t$ liters. The rate of change of salt in solution (measured in $\frac{\text{gm}}{\text{min}}$) is

$$\frac{dS}{dt} = \left(2\frac{\text{gm}}{\text{liter}}\right)\left(4\frac{\text{liters}}{\text{min}}\right) - \left(3\frac{\text{liters}}{\text{min}}\right)\frac{S(t)}{V(t)}\left(\frac{\text{gm}}{\text{liter}}\right) = 8 - \frac{3S(t)}{100+t}\left(\frac{\text{gm}}{\text{min}}\right)$$

b We find an integrating factor first for this first order linear differential equation

$$\ln(u(t)) = \int \frac{3}{100+t}dt = 3\ln(100+t) = \ln(100+t)^3$$

The integrating factor can be taken to be $(100+t)^3$. Using this factor the differential equation becomes simpler, and is integrated easily

$$\frac{d}{dt}\left((1+t)^3 S(t)\right) = 8(1+t)^3, \quad S(t) = 2(1+t) + C(1+t)^{-3}$$

Applying the initial condition $S(0) = 20$, yields

$$S(t) = 2(1+t) + 18(1+t)^{-3}, \quad C(t) = 2 + 18(1+t)^{-4}$$

where $C(t)\left(\frac{\text{gm}}{\text{liter}}\right)$ denotes the concentration of salt in solution.

As $t \to \infty, \quad C(t) \to 2\left(\frac{\text{gm}}{\text{liter}}\right)$.