**Mathematics 237**
**Introduction to differential equations, Fall, 2011**
**Solutions    Homework 2**

**1** Consider the differential equation and intitial value problem

$$6xydx + \big(3x^2 + 4\cos(y)\sin(y)\big)\, dy = 0, \quad y(1) = \frac{\pi}{4}$$

**a)** We rewrite the differential equation

$$\frac{dy}{dx} = \frac{-6xy}{(3x^2 + 4\cos(y)\sin(y))} = f(x,y)$$

$$\frac{\partial f}{\partial y} = \frac{-6x\,(3x^2 + 4\cos(y)\sin(y)) + 24xy\cos(2y)}{(3x^2 + 4\cos(y)\sin(y))^2}$$

The denominator of both $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous functions which do not vanish for the pair of initial values $x = 1, y = \frac{\pi}{4}$. Therefore they dont vanish in some rectangle in the $x - y$ plane which contains this point, and by the existence uniqueness theorem, the initial value problem has a unique solution in some interval containing $x = 1$.
**b)** The equation is exact, since

$$\frac{\partial 6xy}{\partial y} = 6x = \frac{\partial\,(3x^2 + 4\cos(y)\sin(y))}{\partial x}$$

We look for the general solution as the level set of some function $F(x,y)$

$$\begin{aligned}
F(x,y) &= \int 6xydx + h(y) \\
&= 3x^2y + h(y) \\
h'(y) &= \big(3x^2 + 4\cos(y)\sin(y)\big) - 3x^2 \\
&= 2\sin(2y) \\
h(y) &= -\cos(2y) \\
F(x,y) &= 3x^2y - \cos(2y) \\
F(1,\frac{\pi}{4}) &= \frac{3\pi}{4} \qquad (1)
\end{aligned}$$

The unique solution therefore coincides with the level set $3x^2y - \cos(2y) = \frac{3\pi}{4}$.

**2.a)** The right hand side of the differential equation is $f(y) = 2ty^2$. This function is continuous throughout the entire $t-y$ plane. The derivative $\frac{\partial f}{\partial y} = 4ty$ which is continuous throughout the entire $t - y$ plane. The largest rectangle where the conclusions of the existence uniqueness theorem apply would then be the entire $t - y$ plane. The existence uniqueness theorem then states, that for any intial condition $(a, b)$ there is a unique

solution curve in the $t-y$ plane going through that point.
**b)** Separating variables, $\frac{dy}{y^2} = 2tdt$ which integrates easily to $-y^{-1} = t^2 + C$. Solving for $y$ after substiting for the intial values to determine $C$,

$$\begin{aligned}
-b^{-1} &= a^2 + C \\
C &= -b^{-1} - a^2 \\
y &= \frac{-1}{t^2 - b^{-1} - a^2} \\
y(t,a,b) &= \frac{-b}{bt^2 - 1 - ba^2} \\
bt^2 &= 1 + ba^2, \qquad (2)
\end{aligned}$$

where the last equation gives the values at the which the denominator of expression for $y(t)$ is zero. The roots of this quadratic expression determine in general three intervals of the time axis. However only one interval contains the value $t = a$ which is essential to determine the interval of existence of the solution $y(t)$.
**c)** In particular,

$$\begin{aligned}
b = 0 &\Rightarrow t \in \mathbb{R} \\
b > 0 &\Rightarrow \quad -\sqrt{\frac{1+ba^2}{b}} < a < +\sqrt{\frac{1+ba^2}{b}} \\
b > 0 &\Rightarrow \quad -\sqrt{\frac{1+ba^2}{b}} < t < +\sqrt{\frac{1+ba^2}{b}} \\
b < \frac{-1}{a^2} &\Rightarrow \quad t \in \mathbb{R} \\
\frac{-1}{a^2} < b < 0 &\Rightarrow \quad t \in \mathbb{R} \qquad (3)
\end{aligned}$$

This calculation above shows how the interval of existence of the solution $y(t,a,b)$ depends on the values of $a, b$.
**d)** When $a = 0$ from the formula we developed above, $y = \frac{-b}{bt^2-1}$ which is positive when $b > 0$, and $-\sqrt{\frac{1}{b}} < t < +\sqrt{\frac{1}{b}}$. The denominator has roots at $t = \pm\sqrt{\frac{1}{b}}$. The value $t = 0$ must belong to the interval of existence, so this interval is $-\sqrt{\frac{1}{b}} < t < +\sqrt{\frac{1}{b}}$. When $t$ approaches the endpoints of its interval of existence, the value of $y \to +\infty$. This is consistent with the existence uniqueness theorem, since the curve leaves its "box" (which is the infinite $t-y$ plane) as the time parameter approaches the endpoints of the interval of existence.

**3. a)** Multiply the equation by $h(x)$, and check the condition for exactness

$$h(x)\left(3yx^2 + 2xy + y^3\right)dx + h(x)\left(x^2 + y^2\right)dy = 0$$

$$\frac{\partial}{\partial y}h(x)\left(3yx^2+2xy+y^3\right)=\frac{\partial}{\partial x}h(x)\left(x^2+y^2\right)$$
$$h(x)\left(3x^2+2x+3y^2\right)=h'(x)\left(x^2+y^2\right)+h(x)2x$$

$$\begin{aligned} 3h(x)\left(x^2+y^2\right) &= h'(x)\left(x^2+y^2\right) \\ 3h(x) &= h'(x) \\ h(x) &= e^{3x} \text{ (now multiply equation by} e^{3x}\text{)} \\ e^{3x}\left(3yx^2+2xy+y^3\right)dx+e^{3x}\left(x^2+y^2\right) &= 0 \qquad (4) \end{aligned}$$

which is exact as can be checked by usual method.

**b)** We construct the function $F(x,y)$ which satisfies

$$\frac{\partial F}{\partial x}=e^{3x}\left(3yx^2+2xy+y^3\right), \quad \frac{\partial F}{\partial y}=e^{3x}\left(x^2+y^2\right)$$

This can be done by integrating first with respect to $x$ or with respect to $y$. I will integrate with respect to $y$ since this looks much easier....

$$\begin{aligned} F(x,y) &= \int e^{3x}\left(x^2+y^2\right)dy+v(x) \\ &= e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+v(x) \text{ (now differentiate with x)} \\ \frac{\partial F}{\partial x} &= 3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy+v'(x) \text{ (now replace } \frac{\partial F}{\partial x}\text{)} \end{aligned}$$

(6)

$$e^{3x}\left(3x^2y+2xy+y^3\right)=3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy+v'(x)$$

from which we can now observe that $v'(x)=0$. Thus we have the general solution

$$F(x,y)=e^{3x}\left(x^2y+\frac{1}{3}y^3\right)=C$$

**c)** To construct the critical points of the function $F(x,y)$, we need consider the simulaneous equations

$$\frac{\partial F}{\partial x}=3e^{3x}\left(x^2y+\frac{1}{3}y^3\right)+e^{3x}2xy=0$$
$$\frac{\partial F}{\partial y}=e^{3x}\left(x^2+y^2\right)=0$$

The second equation clearly has the only solution $x = y = 0$ which is also a solution of the first equation. Therefore, there is a single critical point of $F$ which is at the origin. The level curves of F can be seen using the Maple command contourplot, or we can apply the second derivative test to see what nature the critical point has. The level curves can be sketched from this information as well. For this we must calculate the discriminant

$$D = \frac{\partial^2 F}{\partial x^2}\frac{\partial^2 F}{\partial y^2} - \left(\frac{\partial^2 F}{\partial x \partial y}\right)^2$$

In this example $D < 0$ which indicates that the origin is a saddle crtical point with two branches of the 0 level curve intersecting at $(0, 0)$ and the contour lines nearby looking like pieces of hyperbolas.