Mathematics 237
Introduction to differential equations, Fall, 2011
Solutions Homework 6

(1) We will compute the solution of

DE $\quad x' = \begin{pmatrix} 0 & 1 \\ -4 & -4 \end{pmatrix} x$ First we calculate the characteristic polynomial of the matrix of coefficients, $\det(A - \lambda I) = -\lambda(-4 - \lambda) + 4 = \lambda^2 + 4\lambda + 4$ The roots of the characteristic polynomial are $\lambda = -2$ with multiplicity two.
Let us find the associated eigenvector V_1. Set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then we must have $AV_1 = -2V_1$, which translates into $y = -2x$. Now choosing $x = 1$ will yield y =-2. Hence

$$V_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Next we look for the generalized eigenvector V_2. The equation giving this vector is $AV_2 = \lambda V_2 + V_1$, which is translates into $y = -2x + 1$. By Choosing $x = 0$, will yield $y = 1$. Hence

$$V_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Therefore the two independent solutions are

$$X^1(t) = e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

,

$$X^2(t) = e^{-2t} \left\{ t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

The general solution will then be

$$X(t) = C_1 e^{-2t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} + C_2 e^{-2t} \left\{ t \begin{pmatrix} 1 \\ -2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

All solutions tend to 0 as $t \to +\infty$ and tend to $+\infty$ as $t \to -\infty$. However the detailed picture of these phase trajectories should include the important information that as $t \to +\infty$, all solutions tend to 0, asymptotic to dominant eigensolution $X_1(t)$. This is also the only solution which evolves as a straight line trajectory in the plane.
The differential equation (DE) is equivalent to a second order homogeneous linear differential equation which governs the motion of a dampened spring mass system. The

governing equation for such systems are $my'' + by' + ky = 0$. If we make a change of variables, $x_1 = y, \;\; x_2 = y'$ then we see that the scalar second order equation is equivalent to $x_1' = x_2, \;\; mx_2' = -kx_1 - bx_2$. When we compare this with the system (DE) we see that $m = 1, b = 4, k = 4$ gives the parameter values which make these two systems equivalent.

(2) We will compute the solution of

$$\text{DE} \quad \begin{pmatrix} x' \\ y' \\ w' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ w \end{pmatrix}$$

First we calculate the characteristic polynomial of the matrix of coefficients,

$$\det\,(A - \lambda I) = (-\lambda - 1)^2(-\lambda + 3)$$

The roots of the characteristic polynomial are $\lambda = -1, 3$.

The corresponding eigenvectors are computed as usual:

$$(A + I)\,\vec{V} = \begin{pmatrix} 2 & 0 & 4 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The corresponding eigenvector is $\vec{V_1} = (-2, 0, 1)^T$.

$$(A - 3I)\,\vec{V} = \begin{pmatrix} -2 & 0 & 4 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The nullspace of this matrix is two dimensional and spanned by two linearly independent eigenvectors $\vec{V}_2 = (0, 1, 0)^T, \vec{V}_3 = (2, 0, 1)^T$.

The general solution is

$$\vec{X}\,(t) = c_1 e^{-t} \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$$

The entire plane of eigensolutions spanned by $\vec{V}_2 = (0, 1, 0)^T, \vec{V}_3 = (2, 0, 1)^T$ is invariant, and expanding as $t \to \infty$. All solutions with initial conditions on the invariant line through $\vec{0}$ and parallel to $\vec{V}_1 = (-2,\ 0\ ,\ 1)^T$ tend to $\vec{0}$ as $t \to \infty$.

The sketch of the phase plane trajectories consists of a two dimensional plane of invariant lines (spanned by $\vec{V}_2 = (0, 1, 0)^T, \vec{V}_3 = (2, 0, 1)^T$) expanding and a transverse invariant line which is contracting as $t \to \infty$. Notice, that we can do this problem without generalised eigenvectors, because the repeating eigenvalue has a two dimensional plane of eigenvectors, which give the complete solution.

(3) We will compute the matrix exponential e^{At} of DE $\quad \mathrm{x}' = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{pmatrix} x$

First, let $\mathrm{A} = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where $\mathrm{B} = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}$, $\quad C = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$.

Then

$$e^{At} = \begin{pmatrix} e^{Bt} & 0 \\ 0 & e^{Ct} \end{pmatrix}$$

which can be seen from the fact that powers of the matrix A (which are used in the construction of e^{At}) have the same block diagonal structure as the matrix A. Our first goal is to find e^{Bt}, e^{Ct}. We will then put these together to form the diagonal block matrix e^{At}. To do this, first we should compute the characteristic polynomials of B and C.
For the matrix B:

$$\det(B - \lambda I) = (1-\lambda)(-1-\lambda) - 3 = \lambda^2 - 4.$$

The roots of B are then $\lambda = 2$ and $\lambda = -2$.
Let us find the associated eigenvector V_1. Set

$$V_1 = \begin{pmatrix} x \\ y \end{pmatrix},$$

then we must have $BV_1 = 2V_1$, which translates into $x + y = 2x$, $3x - y = 2y$. Solving for x and y will imply that $y = x$. Hence

$$V_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore

$$X^1(t) = e^{2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

is a one linearly independent solution for the equation. Next we look for the second vector V_2. We must have $AV_2 = -2V_2$, which translates into $x + y = -2x$, $3x - y = -2y$. Setting $x = 1$, implies that $y = -3$. Hence

$$V_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Therefore

$$X^2(t) = e^{-2t} \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

is a second linearly independent solution.
For the second matrix C:

$$\det(C - \lambda I) = \lambda^2 + 4.$$

The roots of C are then $\lambda = 2i$ and $\lambda = -2i$.
Let us find the associated eigenvector V_3. Set

$$V_3 = \begin{pmatrix} x \\ y \end{pmatrix},$$

then we must have $CV_3 = 2iV_3$, which translates into $-2xi + 2y = 0$, $-2x - 2yi = 0$. Solving for x and y will imply that $y = ix$. Hence

$$V_3 = \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Same steps can be done for the fourth eigenvector:

$$V_4 = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

Hence a fundamental set of solutions is

$$X^3(t) = e^{2it}\begin{pmatrix} 1 \\ i \end{pmatrix}, \qquad X^4(t) = e^{-2it}\begin{pmatrix} i \\ 1 \end{pmatrix}$$

To obtain a set of real valued solutions, we must find the real and imaginary parts of either X^3 or X^4. In fact,

$$X^3(t) = \begin{pmatrix} 1 \\ i \end{pmatrix}(\cos 2t + i\sin 2t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}.$$

Consequently, the set of real valued solutions is

$$X^3(t) = \begin{pmatrix} \cos 2t \\ -\sin 2t \end{pmatrix}, \qquad X^4(t) = \begin{pmatrix} \sin 2t \\ \cos 2t \end{pmatrix}$$

Therefore, $\mathrm{X(t)} = \begin{pmatrix} e^{2t} & e^{-2t} & 0 & 0 \\ e^{2t} & -3e^{-2t} & 0 & 0 \\ 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & \cos 2t \end{pmatrix}$ is a fundamental matrix solution of the differential equation given in the question.

We now compute $\mathrm{X}^{-1}(0) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then

$$e^{At} = X(t)X^{-1}(0) = \begin{pmatrix} e^{2t} & e^{-2t} & 0 & 0 \\ e^{2t} & -3e^{-2t} & 0 & 0 \\ 0 & 0 & \cos 2t & \sin 2t \\ 0 & 0 & -\sin 2t & \cos 2t \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(2)

(4) We will compute the fundamental matrix solution of DE $\quad \mathrm{x}' = \begin{pmatrix} 2 & -1 & -4 \\ 0 & 2 & -4 \\ 0 & 1 & -2 \end{pmatrix} x$

First we calculate the characteristic polynomial of the matrix of coefficients, $\det(A - \lambda I) = (2-\lambda)(\lambda^2 - 4 + 4) = \lambda^2(2-\lambda)$ The roots of the characteristic polynomial are $\lambda = 0$ with multiplicity two and $\lambda = 2$ with multiplicity 1.
Set

$$V_1 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Then we must have $AV_1 = 0$, which translates into $x_1 = 3x_3$, $x_2 = 2x_3$. Setting $x_3 = 1$, implies that

$$V_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

Consequently,

$$X^1(t) = e^{0t} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

is one solution of the differential equation. Next we look for the generalized eigenvector V_2. Set

$$V_2 = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

The equation giving this vector is $AV_2 = \lambda V_2 + V_1$, which is translates into $y_2 = 2y_3 + 1$ and $2y_1 = y_2 + 4y_3 + 3$. From the first translated equation, if $y_3 = 0$ then that would imply $y_2 = 1$ and using the values of y_2 and y_3 in the second translated equation imply that $y_1 = 2$. Hence

$$V_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

and

$$X^2(t) = t \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3t+2 \\ 2t+1 \\ t \end{pmatrix}$$

is a second linearly independent solution. Finally, we look for the corresponding eigenvector V_3. Set

$$V_3 = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

. The equation giving this vector is $AV_3 = 2V_3$, which translates into $z_3 = 0$, $z_2 = z_3 = 0$ and z_1 is arbitrary.

$$V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Hence

$$X^3(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is a third linearly independent solution. Therefore,

$$X(t) = \begin{pmatrix} 3 & 3t+2 & e^{2t} \\ 2 & 2t+1 & 0 \\ 1 & t & 0 \end{pmatrix}$$

is a fundamental matrix solution of the differential equation.