Mathematics 280

Advanced Calculus, Fall 2012

Solutions to Homework 1

1. (i) The vertical component of the velocity is 7.5 km/hour. Thus the speed in the vertical direction is 7.5 km/hour.

(ii) The horizontal component of the velocity vector is (75,150)= 75(1,2). The length of this vector, which is the horizontal speed is $\|(75, 150)\| = 75\sqrt{5}$ km/hour. The horizontal distance to the skyscaper is $\|(5, 10)\| = 5\|(1, 2)\| = 5\sqrt{5}$ km. Thus using the formula (for constant speed) distance = velocity times time, we see that the time required to arrive over the skyscraper is

$$\text{time to arrive over skyscraper} = \frac{5\sqrt{5}}{75\sqrt{5}} = \frac{1}{15} \text{ hours}$$

(iii) Since we know the vertical speed, and the time to arrival, we can calulate the vertical distance of the airplane as it flies over the skyscraper. This distance is $\frac{7.5}{15} = \frac{1}{2}$ km. Thus the height of the airplane above the skyscraper is 500 - 150= 350 meters.

2. In this problem, the main point is to decide what is the parameter which we should use to describe the rotation of the reflector on the wheel. Thinking about this for a moment, and drawing a picture, we are confronted with the problem of deciding what the no slip condition of the wheel on the road should mean.

Let us set some coordinates in the plane. The usual x, y rectangular coordinates with x as the horizontal distance and y as the vertical height of a point in the plane. We see that the no slip condition amounts to asking that the horizontal distance along the x-axis should be exactly the same as the arclength of the wheel (with radius a) rolling along this axis. This arclength is s, and the corresponding angle for a wheel is $\theta = \frac{t}{a}$, which is measured clockwise from the postion when $t = 0$.

Let us use this horizontal distance as the parameter t. That is $t = x$ where x is the horizontal component of the center of the wheel as it moves along the x-axis. We assume that initially, the wheel is aligned so that the point of contact P is at the origin of the coordinate system. We want to describe the motion of the moving point P(t).

We do this by utilizing the parallelogram rule for vector addition. The point P(t) can be realized by adding three vectors, which are horizontal, to describe the horizontal distance moved by the wheel, a constant vertical vector to describe the position of the center, and a vector which parameterizes points on the wheel in terms of the angle θ

$$\vec{OP}(t) = s\mathbf{i} + a\mathbf{j} + (-b\sin(t/a), -b\cos(t/a))$$

Decomposing the vector sum into components we get

$$x(t) = t - b\sin(t/a), \qquad y(t) = a - b\cos(t/a)$$

3(a). Taking inner product of the two vectors gives

$$\begin{aligned}
\langle \|y\|x + \|x\|y, \|y\|x - \|x\|y \rangle &= \langle \|y\|x, \|y\|x \rangle - \langle \|y\|x, \|x\|y \rangle + \langle \|x\|y, \|y\|x \rangle - \langle \|x\|y, \|x\|y \rangle \\
&= \|y\|^2\|x\|^2 - \|y\|\|x\|\langle x, y \rangle + \|x\|\|y\|\langle y, x \rangle - \|x\|^2\|y\|^2 \\
&= 0
\end{aligned}$$

(b) The vectors given in part (a), are the diagonals of the parallelogram spanned by $\|y\|x$ and $\|x\|y$. The angle between these vectors is the same as the angle between x, y since this angle does not change by scaling the sides of the parallelogram. However by this scaling, we have arranged for the parallelogram spanned by $\|y\|x$ and $\|x\|y$ to be rhomboid. That is the angle between the sides $\|y\|x$ and $\|x\|y$ is bisected by the diagonal $\|y\|x + \|x\|y$,