Mathematics 280

Advanced Calculus, Fall 2016

Homework 6, due Friday November 4, by NOON!

1(a) Set up a path integral, and compute the work done by the force $\tilde{\mathbf{F}} = \left(x^2 - y^2\right)\mathbf{i} + 2xy\mathbf{j}$ by moving a particle of mass m, along the square in the plane, bounded by the coordinate axes and lines $x = 3, y = 3$ with the counterclockwise orientation.

We need to set up four separate paths, which are joined together as the PWS path described in the problem.

$$\begin{aligned}
\mathbf{r}_1(t) &= (t, 0), 0 \leq t \leq 3 \\
\mathbf{r}_2(t) &= (3, t), 0 \leq t \leq 3 \\
\mathbf{r}_3(t) &= (3 - t, 3), 0 \leq t \leq 3 \\
\mathbf{r}_4(t) &= (0, 3 - t), 0 \leq t \leq 3
\end{aligned}$$

On each path we compute $\tilde{\mathbf{F}} \cdot \mathbf{dS} = \tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}dt$ (since t denotes arclength on each segment of the path, and $\tilde{\mathbf{T}}$ denotes the unit tangent vector along that path)

$$\begin{aligned}
\tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}_\mathbf{1} &= t^2 \\
\tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}_\mathbf{2} &= 6t \\
\tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}_\mathbf{3} &= 9 - (3 - t)^2 \\
\tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}_\mathbf{4} &= 0
\end{aligned}$$

The calculation of the work done by the vector field $\tilde{\mathbf{F}}$ along the entire path then can be summed in the following way

$$\text{Work Done } = \int_C \tilde{\mathbf{F}} \cdot \tilde{\mathbf{T}}ds = \int_0^3 \left[t^2 + 6t + 9 - (3 - t)^2 + 0\right] dt = \frac{1}{3}27 + 3(9) + 9(3) - \frac{1}{3}27$$

(b) Compute the path integral $\int_C \tilde{\mathbf{F}} \cdot \mathbf{dS}$ where $\tilde{\mathbf{F}} = \left(x^2 - y^2\right)\mathbf{i} + x\mathbf{j}$ and $\mathbf{C}$ is one circuit of the circle $x^2 + y^2 = 4$ in the counterclockwise direction.

Let $r(t) = (2\cos(t), 2\sin(t)), 0 \leq t \leq 2\pi$ give a parameterization of the circular path. Then $r'(t) =$

$(-2\sin(t), 2\cos(t))$ and the unit tangent vector is $\tilde{\mathbf{T}} = (-\sin(t), \cos(t))$.

$$\begin{aligned}
\text{W} &= \int_C \tilde{\mathbf{F}} \cdot \mathbf{dS} \\
&= \int_0^{2\pi} \left((2\cos(t))^2 - (2\sin(t))^2\right)(-2\sin(t))dt + \int_0^{2\pi} (2\cos(t))(2\cos(t))dt \\
&= \int_0^{2\pi} \left(4\sin^3(t) + 4\cos^2(t)\right) dt \qquad \left(\int_0^{2\pi} \cos^2(t)\sin(t)dt = 0\right) \\
&= 4\left[2\pi + \pi\right]
\end{aligned}$$

2(a) Is there a vector field $\tilde{\mathbf{F}}$ so that Curl$(\tilde{\mathbf{F}}) = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$? Explain.

Calculating the divergence of the vector field $\tilde{\mathbf{G}} = xy^2\mathbf{i} + yz^2\mathbf{j} + zx^2\mathbf{k}$ we find div$\tilde{\mathbf{G}} = x^2 + y^2 + z^2 \neq 0$. Therefore $\tilde{\mathbf{G}}$ cannot be a curl field, since div curl $= 0$. Therefore there is no such vector field $\tilde{\mathbf{F}}$

(b) Is there a vector field $\tilde{\mathbf{F}}$ so that Curl$(\tilde{\mathbf{F}}) = 2\mathbf{i} + 1\mathbf{j} + 3\mathbf{k}$? If so, find one.

Let $\tilde{\mathbf{G}} = 2\mathbf{i} + 1\mathbf{j} + 3\mathbf{k}$. Since div $(2\mathbf{i} + 1\mathbf{j} + 3\mathbf{k}) = 0$ we are led to look for a field $\tilde{\mathbf{F}}$ with curl $\tilde{\mathbf{F}} = \tilde{\mathbf{G}}$. As we did in class, we consider the radial vector field $\tilde{\mathbf{r}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\tilde{\mathbf{v}} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ for undetermined constants a,b,c.

We compute the cross product $\vec{\mathbf{v}} \times \vec{\mathbf{r}} = (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k}$ and then take the curl of this expression

$$\text{curl } (bz - cy)\mathbf{i} - (az - cx)\mathbf{j} + (ay - bx)\mathbf{k} = 2a\mathbf{i} + 2b\mathbf{j} + 2c\mathbf{k} = 2\mathbf{i} + 1\mathbf{j} + 3\mathbf{k}$$

which requires that $a = 1, b = \frac{1}{2}, c = \frac{3}{2}$

3.Consider the vector field $\vec{\mathbf{F}} : \mathbb{R} \times (0, \infty) \to \mathbb{R}^2$ given by

$$\vec{\mathbf{F}}(x, y) = \frac{x + xy^2}{y^2}\mathbf{i} - \frac{x^2 + 1}{y^3}\mathbf{j}$$

a) Determine if $\vec{\mathbf{F}}$ has the path independent property for every oriented path in the domain of $\vec{\mathbf{F}}$.

Compute the curl of the given vector field

$$\text{curl } \vec{\mathbf{F}} = \frac{\partial \vec{\mathbf{F}}_2}{\partial x} - \frac{\partial \vec{\mathbf{F}}_1}{\partial y} = -\frac{2x}{y^3} - \frac{-2x}{y^3} = 0$$

Since curl is zero, to conclude that the vector has the path independence property, we should consider the property of the doman being simply connected (without this property, the path independence does not hold!) . We examine the vector field $\vec{\mathbf{F}}(x, y) = \frac{x+xy^2}{y^2}\mathbf{i} - \frac{x^2+1}{y^3}\mathbf{j}$ and see that the domain consists of all the points in the x-y plane where $y > 0$. The other part of the doman where $y < 0$ will not factor into the discussion and does not contain the path joining the given points. The upper half plane where $y > 0$ is simply connected and as we have seen curl $\vec{\mathbf{F}} = 0$ which guarantees that the vector field has the path independence property.

b) Find the work done by $\vec{\mathbf{F}}$ in moving a particle along the curve $y = 1 + x - x^2$ from (0,1) to (1,1).

The work done is independent of the path chosen to move the particle between the endpoints at (0,1), (1,1). So we can choose any path which connects these two points, and the simplest such path is the straight line from (0,1) to (1,1). This has parameterization $\vec{\mathbf{r}}(t) = (t, 1), 0 \leq t \leq 1$.

$$\begin{aligned} \text{W} &= \int_C \tilde{\mathbf{F}} \cdot \mathbf{dS} \\ &= \int_0^1 2t dt = 1(\text{ units N-m}) \end{aligned}$$