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Student number

QUEEN'S UNIVERSITY

FACULTY OF ARTS AND SCIENCE

DEPARTMENT OF MATHEMATICS AND STATISTICS

MATH 280 FINAL EXAMINATION

DECEMBER 2011

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• Answer all questions on these pages. When asked to "explain", "prove", or "show", do so in full sentences. When quoting a theorem to justify your work, make sure to verify that the hypotheses of the theorem are satisfied as part of your justification.

• **CLOSED BOOK EXAM. NO CALCULATORS ALLOWED.**

• **PLEASE NOTE:** Proctors are unable to respond to queries about the interpretation of exam questions. Do your best to answer exam questions as written.

Scoring:	Part I:	________	(out of 10)
	Part II:	________	(out of 10)
	Part III:	________	(out of 10)
	Part IV:	________	(out of 10)
	Part V:	________	(out of 10)
	Part VI:	________	(out of 10)
	Part VII:	________	(out of 10)
	Part VIII:	________	(out of 5)
	Part IX:	________	(out of 10)
	Part X:	________	(out of 10)
	Total:	________	(out of 95)

Part I. Let

$$f(x,y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

(a) Find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

(b) Calculate the directional derivative $D_{\vec{u}}f(0,0)$, where $\vec{u} = \frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$.

(c) Is it true that $D_{\vec{u}}f(0,0) = \vec{\nabla} f(0,0) \cdot \vec{u}$? Is $f(x,y)$ differentiable at $(0,0)$? Explain!

Part II. Let $D \subset \mathbb{R}^2$ be the region in the first quadrant bounded by the lines $x = y$ and $x = 3y$ and the hyperbolas $xy = 1$ and $xy = 2$.

(a) Consider the change of variables $u = \frac{x}{y}$, $v = xy$. Express x and y as functions of u and v and calculate $\frac{\partial(x,y)}{\partial(u,v)}$.

(b) Using the change of variables from (a), find the area of D.

Part III. Evaluate the iterated integral

$$\int_0^{\pi} \int_y^{\pi} \frac{\sin x}{x} \, dx dy.$$

Hint. Change the order of integration.

Part IV. Calculate the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ over the unit ball $B : x^2 + y^2 + z^2 \leq 1$.

Part V.

A. Evaluate the line integral $\int_C xz\,dx + z\,dy + \sqrt{y}\,dz$, where C is the curve parametrized by $\vec{\gamma}(t) = -t\,\vec{i} + t^2\,\vec{j} + 3t\,\vec{k}$ for $0 \le t \le 2$.

B. Let $\vec{F} = z\cos(xz+y^2)\,\vec{i} + 2y\cos(xz+y^2)\,\vec{j} + x\cos(xz+y^2)\,\vec{k}$. Calculate $\int_C \vec{F}\cdot d\vec{r}$, where C is the curve parametrized by $\vec{\gamma}(t) = t\,\vec{i} + \sin(2t)\,\vec{j} + \sin(t)\,\vec{k}$ for $0 \le t \le \frac{\pi}{2}$.

Part VI.

A. Calculate the flux of $\vec{F} = x\,\vec{j} + z\,\vec{k}$ through the surface S parametrized by $\vec{\sigma} : [0,1] \times [0,2] \to \mathbb{R}^3$ where $\vec{\sigma}(u,v) = (u^2 + 2v)\,\vec{i} + v^2\,\vec{j} + 3u\,\vec{k}$ and oriented downwards.

B. Calculate the flux of $\vec{G} = (e^{x^2} + z)\,\vec{i} + (e^z \sin y - x^2 y)\,\vec{j} + (e^{x^2} + z - 1)\,\vec{k}$ through the disk D of radius 3 centred at $(9, 3, -2)$ and with normal vector $\vec{n} = -\vec{i} + \vec{k}$.

Part VII. Let $\vec{F} = (xy^2 + \sin(yz))\,\vec{i} + (yz^2 + \sin(zx))\,\vec{j} + (zx^2 + \sin(xy))\,\vec{k}$.

(a) Calculate the flux of $\vec{F}$ through the boundary of the cube with vertices $(\pm 1, \pm 1, \pm 1)$, i.e., the cube with side length 2 centred at the origin and edges parallel to the coordinate axes.

(b) Will the flux change if we rotate the cube from (a) about the origin?

Hint. For (b) you do not need to calculate anything. Instead think of a geometric argument.

Part VIII.

State Stokes theorem. Carefully explain what orientations are involved.

Part IX.

A. Show that $\vec{F} = (y^2 + 3x^2z^2)\,\vec{i} + (2xy + 4yz^2)\,\vec{j} + (2x^3z + 4y^2z)\,\vec{k}$ is path-independent.

B. Does the vector field $\vec{G} = \cos(y+z)\,\vec{i} + (x^3y - y^2z^2)\,\vec{j} + (-x^3z + y^2z^3)\,\vec{k}$ have a vector potential?

Part X. Let $U = \mathbb{R}^2 \backslash \{(0,0)\}$ and let $\vec{F} = F_1\vec{i} + F_2\vec{j} : U \to \mathbb{R}^2$ be a smooth vector field such that $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$.

(a) Prove that $\vec{F}$ is a gradient field if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$, where C is the unit circle oriented counter-clockwise.

(b) Prove that there exists a constant $c \in \mathbb{R}$ such that

$$\vec{G} = \vec{F} + c\,\frac{-y\,\vec{i} + x\,\vec{j}}{x^2 + y^2}$$

is a gradient field.