Mathematics 280

Advanced Calculus, Fall 2016

Solutions to Homework 2

(Q1) Part (a): Notice that $|ab| = |a||b|, a^2 = |a|^2$, and $b^2 = |b|^2$. Thus,

$$\begin{aligned} 2|ab| \le a^2 + b^2 &\Leftrightarrow 2|a||b| \le |a|^2 + |b|^2 \\ &\Leftrightarrow 0 \le |a|^2 + |b|^2 - 2|a||b| \\ &\Leftrightarrow 0 \le (|a| - |b|)^2 \end{aligned}$$

which is clearly true for all real numbers a, b, since the right hand side cannot be negative.

Part (b): Let $0 < \|(x, y)\| < \delta$, so that $\sqrt{x^2 + y^2} < \delta$, which is equivalent to $x^2 + y^2 < \delta^2$. Keeping this in mind, we have

$$\begin{aligned} |f(x, y)| &= \left| \frac{xy(x^2 - y^2)}{x^2 + y^2} \right| \\ &= \left| \frac{xy}{x^2 + y^2} \right| |x^2 - y^2| \\ &= \frac{|xy|}{x^2 + y^2} |x^2 - y^2| \\ &\le \frac{|xy|}{x^2 + y^2} (|x^2| + |y^2|) \text{ by the triangle inequality.} \end{aligned}$$

Applying part (a), we see that $|xy| \le \frac{x^2+y^2}{2}$ and so $\frac{|xy|}{x^2+y^2} \le \frac{1}{2}$. Since $|x^2| + |y^2| = x^2 + y^2 < \delta^2$, we have $|f(x, y)| < \frac{\delta^2}{2}$.

It is clear that $f(x, y)$ is continuous at any $(x, y) \neq (0, 0)$. To conclude that $f(x, y)$ is continuous on its domain, we must check continuity at $(x, y) = (0, 0)$. Namely, let $\epsilon > 0$ and set $\delta = \sqrt{2\epsilon}$. Suppose that (x, y) satisfies $\|(x, y) - (0, 0)\| = \|(x, y)\| < \delta$. By the previous calculation, $|f(x, y)| < \frac{\delta^2}{2} = \frac{2\epsilon}{2} = \epsilon$, and we are done.

Part (c): We compute $f_x(x, y)$ first. By definition,

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(0 + h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{f(h, 0) - 0}{h} = \lim_{h \to 0} \frac{\frac{0}{h^2}}{h} = \lim_{h \to 0} 0 = 0.$$

Away from $(0,0)$ we may apply the rules of differentiation. Letting $\partial_x = \frac{\partial}{\partial x}$, we have

$$
\begin{aligned}
f_x(x,y) &= \frac{\Big(\partial_x\big(xy(x^2-y^2)\big)\Big)(x^2+y^2) - xy(x^2-y^2)\big(\partial_x(x^2+y^2)\big)}{(x^2+y^2)^2} \\
&= \frac{\big(y(x^2-y^2)+xy(2x)\big)(x^2+y^2) - xy(x^2-y^2)(2x)}{(x^2+y^2)^2} \\
&= \frac{(3x^2y-y^3)(x^2+y^2) - 2x^2y(x^2-y^2)}{(x^2+y^2)^2} \\
&= \frac{(3x^4y - x^2y^3 + 3x^2y^3 - y^5) - 2x^4y + 2x^2y^3}{(x^2+y^2)^2} \\
&= \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \\
&= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2+y^2)^2} \\
&= \frac{y\big((x^2+y^2)(x^2-y^2) + 4x^2y^2\big)}{(x^2+y^2)^2} \\
&= \frac{y(x^2-y^2)}{(x^2+y^2)} + \frac{4x^2y^3}{(x^2+y^2)^2}.
\end{aligned}
$$

By a similar computation to above, we find $f_y(0,0) = 0$. To get $f_y(x,y)$ at other points, notice that $f(x,y) = -f(y,x)$. Thus, $f_y(x,y) = \partial_y\big(-f(y,x)\big) = -f_y(y,x)$. In other words, we can get $f_y(x,y)$ from the big computation we just did, by switching the role of x and y, and multiplying by -1. We end up with

$$
\begin{aligned}
f_y(x,y) &= -\frac{x(y^2-x^2)}{(x^2+y^2)} - \frac{4y^2x^3}{(x^2+y^2)^2} \\
&= \frac{x(x^2-y^2)}{(x^2+y^2)} - \frac{4x^3y^2}{(x^2+y^2)^2}.
\end{aligned}
$$

Part (d): From the expressions given above, we see that $f_x(x,y)$ and $f_y(x,y)$ are continuous away from $(0,0)$. We check continuity at $(0,0)$.

Let $\epsilon > 0$ and set $\delta = \sqrt{\frac{\epsilon}{2}}$. Then, for any $0 < \|(x, y)\| < \delta$ we have $x^2 + y^2 < \frac{\epsilon}{2}$, and we compute

$$\begin{aligned}
|f_x(x,y)| &= \left|\frac{y(x^2-y^2)}{(x^2+y^2)} + \frac{4x^2y^3}{(x^2+y^2)^2}\right| \\
&\leq \left|\frac{y(x^2-y^2)}{(x^2+y^2)}\right| + \left|\frac{4x^2y^3}{(x^2+y^2)^2}\right| \\
&\leq \left|\frac{y}{(x^2+y^2)}\right|(|x^2|+|y^2|) + \left|\frac{4x^2y^3}{(x^2+y^2)^2}\right| \\
&= |y| + 4|y|\left|\frac{xy}{x^2+y^2}\right|^2 \\
&\leq |y| + 4|y|(\frac{1}{2})^2 \text{ by part (a)} \\
&= 2|y| \\
&\leq 2(x^2+y^2) \\
&< 2\delta^2 = 2(\frac{\epsilon}{2}) = \epsilon.
\end{aligned}$$

Hence, we see that $f_x(x, y)$ is continuous at $(0, 0)$. The same method shows that $f_y(x, y)$ is also continuous at $(0, 0)$. Since the partial derivatives of f exist *and are continuous*, we may conclude that f is differentiable at $(0, 0)$.

Part (e): We compute from the definition

$$\begin{aligned}
f_{xy}(0,0) &= \lim_{h\to 0} \frac{f_x(0,0+h) - f_x(0,0)}{h} \\
&= \lim_{h\to 0} \frac{f_x(0,h) - 0}{h} \\
&= \lim_{h\to 0} \frac{\frac{h(0^2-h^2)}{(0^2+h^2)} + \frac{4(0)^2h^3}{(0^2+h^2)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{-h^3}{h^2} + 0}{h} \\
&= \lim_{h\to 0} \frac{-h}{h} = -1.
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
f_{yx}(0,0) &= \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} \\
&= \lim_{h\to 0} \frac{f_y(h,0) - 0}{h} \\
&= \lim_{h\to 0} \frac{\frac{h(h^2-0^2)}{(h^2+0^2)} - \frac{4h^3 0^2}{(h^2+0^2)^2}}{h} \\
&= \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1.
\end{aligned}$$

(Q2) We wish to show that $x^2 \frac{\partial p}{\partial x} = y^2 \frac{\partial p}{\partial y}$. Denote $F(x,y) = \frac{x+y}{xy}$, which defines a differentiable function $F : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$, and gives $p = q \circ F$. We compute

$$\begin{gathered}
\nabla p = q'(F(x,y)) \cdot \nabla F \text{ by the chain rule, which is equivalent to} \\
\begin{pmatrix} \frac{\partial p}{\partial x} & \frac{\partial p}{\partial y} \end{pmatrix} = q'(F(x,y)) \cdot \begin{pmatrix} \frac{-1}{x^2} & \frac{-1}{y^2} \end{pmatrix},
\end{gathered}$$

which shows that $x^2 \frac{\partial p}{\partial x} = -q'(F(x,y)) = y^2 \frac{\partial p}{\partial y}$.