Mathematics 280

Advanced Calculus, Fall 2016

Solutions to Homework 4

1(a) we first compute $q = F(1,1) = (e, 0, 1)$.

(b) We write out the composite function $G \circ F$ in all of its (ugly) glory.

$$G \circ F(x,y) = G(F(x,y)) = \Big(\cos(y \sin(\pi x) e^{x^2}), xy - e^{2x^2}\Big)$$

This is a vector mapping from $\mathbb{R}^2$ into $\mathbb{R}^2$ so its derivative will be a 2x2 matrix with entries

$$\begin{bmatrix} \frac{\partial (G\circ F)_1}{\partial x} & \frac{\partial (G\circ F)_1}{\partial y} \\ \frac{\partial (G\circ F)_2}{\partial x} & \frac{\partial (G\circ F)_2}{\partial y} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 - 4e^2 & 1 \end{bmatrix}$$

after having evaluated the partial derivatives at (1,1).

(c) The matrices DF(1,1) and DG(q) are 3 by 2, and 2 by 3 matrices respectively with entries

$$DF(1,1) = \begin{bmatrix} \frac{\partial (F)_1}{\partial x} & \frac{\partial (F_1}{\partial y} \\ \frac{\partial (F)_2}{\partial x} & \frac{\partial (F)_2}{\partial y} \\ \frac{\partial (F)_3}{\partial x} & \frac{\partial (F)_3}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xe^{x^2} & 0 \\ y\pi \cos(\pi x) & \sin(\pi x) \\ y & x \end{bmatrix} = \begin{bmatrix} 2e & 0 \\ -\pi & 0 \\ 1 & 1 \end{bmatrix}$$

$$DG(q) = \begin{bmatrix} \frac{\partial (G_1}{\partial u} & \frac{\partial (G_1}{\partial v} & \frac{\partial (G_1}{\partial w} \\ \frac{\partial (G_2}{\partial u} & \frac{\partial (G_2}{\partial v} & \frac{\partial (G_2}{\partial w} \end{bmatrix} = \begin{bmatrix} -v\sin(uv) & -u\sin(uv) & 0 \\ -2u & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2e & 0 & 1 \end{bmatrix}$$

(d) The matrix product is

$$DG(1)DF(1,1) = \begin{bmatrix} 0 & 0 \\ 1 - 4e^2 & 1 \end{bmatrix}$$

which agrees exactly with the earlier computation for $D(G \circ F)(1,1)$. This computation confirms the chain rule in this particular case.

2(a) Using polar coordinate with the vector function $\mathbf{F}(r,\theta) = (r\cos(\theta), r\sin(\theta))$, we have

$$\mathbf{DF}(r,\theta) = \begin{bmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{bmatrix}$$

We compute first $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Using this we calculate

$$\mathbf{F}\left(2, \frac{\pi}{3}\right) = (1, \sqrt{3}), \qquad \mathbf{DF}\left(2, \frac{\pi}{3}\right) = \begin{bmatrix} \frac{1}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

2(b) The chain rule gives the formula , with $\cdot$ denoting the matrix product

$$\mathbf{DH}\left(2, \frac{\pi}{3}\right) = \mathbf{DG}(1, \sqrt{3}) \cdot \mathbf{DF}\left(2, \frac{\pi}{3}\right) = \left[\frac{\partial G}{\partial x}, \frac{\partial G}{\partial y}\right] \cdot \begin{bmatrix} \frac{1}{2} & -\sqrt{3} \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

(c) From the preceeding computation we can invert the matrix for $\mathbf{DF}\left(2, \frac{\pi}{3}\right)$ to get

$$\mathbf{DG}\left(1, \sqrt{3}\right) = \mathbf{DH}\left(2, \frac{\pi}{3}\right) \cdot \mathbf{DF}^{-1}\left(2, \frac{\pi}{3}\right) = \left[\frac{\partial H}{\partial r}, \frac{\partial H}{\partial \theta}\right] \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{4} & \frac{1}{4} \end{bmatrix}$$