1 (6.2.5, p. 388). *(a) Use Green's theorem to calculate the line integral*

$$\oint_C y^2 dx + x^2 dy,$$

where C is the path formed by the square with vertices $(0,0)$, $(1,0)$ $(0,1)$ *and* $(1,1)$ *oriented counterclockwise.*

(b) Verify your answer to part (a) by calculating the line integral directly.

Solution. (a) Let D be the square enclosed by the path C. By Green's theorem, we have

$$\begin{aligned}\oint_C y^2 dx + x^2 dy &= \iint_D \left(\frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial y}\right) dA = \iint_D (2x - 2y)\, dA \\ &= \int_0^1 \int_0^1 (2x-2y)\, dxdy = \int_0^1 \left[x^2 - 2xy\right]_0^1 dy = \int_0^1 1 - 2y\, dy = 0.\end{aligned}$$

(b) Let C_1, C_2, C_3 and C_4 be the four sides of the square traversed counterclockwise, with C_1 being the side joining $(0,0)$ and $(1,0)$. Write $\mathbf{F}(x,y) = (y^2, x^2)$. We can parametrize the C_i as follows

i	$\mathbf{x_i}(t)$	$\mathbf{x_i'}(t)$	$\mathbf{F}(\mathbf{x_i}(t))$	$\mathbf{F}(\mathbf{x_i}(t)) \cdot \mathbf{x_i'}(t)$
1	$(t,0)\,(0 \le t \le 1)$	$(1,0)$	$(0,t^2)$	0
2	$(1,t)\,(0 \le t \le 1)$	$(0,1)$	$(t^2,1)$	1
3	$(1-t,1)\,(0 \le t \le 1)$	$(-1,0)$	$(1,(1-t)^2)$	-1
4	$(0,1-t)\,(0 \le t \le 1)$	$(0,-1)$	$((1-t)^2,0)$	0

Then

$$\oint_C y^2 dx + x^2 dy = \sum_{i=1}^{4} \int_{C_i} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 0 + 1 - 1 + 0\, dt = 0,$$

as before.

2 (6.2.7, p. 389). *Evaluate*

$$\oint_C (x^2 - y^2)\, dx + (x^2 + y^2)\, dy,$$

where C is the boundary of the square with vertices $(0,0)$, $(1,0)$ $(0,1)$ *and* $(1,1)$ *oriented* clockwise. *Use whatever method of evaluation seems appropriate.*

Solution.

Method 1 (Brute force calculation). Let C_1, C_2, C_3 and C_4 be the four sides of the square traversed *clockwise*, with C_1 being the side joining $(0,0)$ and $(0,1)$. Write $\mathbf{F}(x,y) = (x^2 - y^2, x^2 + y^2)$. We can parametrize the C_i as follows

i	$\mathbf{x_i}(t)$	$\mathbf{x_i'}(t)$	$\mathbf{F}(\mathbf{x_i}(t))$	$\mathbf{F}(\mathbf{x_i}(t)) \cdot \mathbf{x_i'}(t)$
1	$(0,t)\,(0 \le t \le 1)$	$(0,1)$	$(-t^2,t^2)$	t^2
2	$(t,1)\,(0 \le t \le 1)$	$(1,0)$	(t^2-1,t^2+1)	t^2-1
3	$(1,1-t)\,(0 \le t \le 1)$	$(0,-1)$	$(1-(1-t)^2, 1+(1-t)^2)$	$-1-(1-t)^2$
4	$(1-t,0)\,(0 \le t \le 1)$	$(-1,0)$	$((1-t)^2,(1-t)^2)$	$-(1-t)^2$

Then

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = \sum_{i=1}^{4}\int_{C_i}\mathbf{F}\cdot d\mathbf{s} = \int_0^1 \left(t^2+t^2-1-1-(1-t)^2-(1-t)^2\right) dt$$
$$= \int_0^1 \left(2t^2-2-2(1-2t+t^2)\right) dt = \int_0^1 (-4+4t)\, dt = 4\left(-1+\frac{1}{2}\right) = -2.$$

Method 2 (Green's theorem). Because the path C is oriented clockwise, we cannot immediately apply Green's theorem, as the region bounded by the path appears *on the right-hand side* as we traverse the path C (*cf.* the statement of Green's theorem on p. 381). However, we know that if we let $\mathbf{x}$ be a clockwise parametrization of C and $\mathbf{y}$ an *orientation-reversing (that is, counterclockwise) reparametrization*, then we have

$$\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s}.$$

(*cf.* theorem 1.5, p. 371 — the proof involves simply the single-variable chain-rule). Now, letting $\tilde{C}$ be the path C with the counterclockwise orientation and D be the square enclosed by C (and $\tilde{C}$), we have by Green's theorem

$$\oint_{\tilde{C}} \mathbf{F}\cdot d\mathbf{s} = \iint_D \left(\frac{\partial(x^2+y^2)}{\partial x} - \frac{\partial(x^2-y^2)}{\partial y}\right) dA = \iint_D (2x+2y)\ dA$$
$$= \int_0^1\int_0^1 (2x+2y)\, dx dy = \int_0^1 \left[x^2+2xy\right]_0^1 dy = \int_0^1 1+2y\, dy = 2.$$

Thus,

$$\oint_C \mathbf{F}\cdot d\mathbf{s} = -\oint_{\tilde{C}} \mathbf{F}\cdot d\mathbf{s} = -2,$$

as before.

3 (6.2.8, p. 389)**.** *Use Green's theorem to find the work done by the vector field*

$$\mathbf{F} = (4y-3x)\mathbf{i} + (x-4y)\mathbf{j}$$

on a particle as the particle moves counterclockwise once around the ellipse $x^2+4y^2=4$.

Solution. Let C be the boundary of the ellipse, oriented counterclockwise, and E be the ellipse. Applying Green's theorem, we have

$$W = \oint_C (4y-3x)\, dx + (x-4y)\, dy = \iint_E (1-4)\, dA = -3\iint_E dA = -3\times \text{Area}(E).$$

But the area of an ellipse with the semi-major and semi-minor axes having lengths a and b, respectively, is $ab\pi$. Thus, $W = -3(2\cdot 1\cdot \pi) = -6\pi$.