1 (7.1.7, p. 418)**.** *Let S be the surface parametrized by $x = s\cos t$, $y = s\sin t$, $z = s^2$, where $s \geq 0$, $0 \leq t \leq 2\pi$.*

(a) At what points is S smooth? Find an equation for the tangent plane at thet point $(1, \sqrt{3}, 4)$.

(b) Describe S by an equation of the form $z = f(x, y)$.

(c) Using your answer in part (c), discuss whether S has a tangent plane at every point.

Solution. (a) Write $\mathbf{X}(s,t) = (s\cos t, s\sin t, s^2)$. It is clear that $\mathbf{X}$ is everywhere infinitely differentiable as a function $\mathbb{R}^2 \to \mathbb{R}^3$, so to check the points where S is smooth we only need to find points where the normal vector to S given by $\frac{\partial \mathbf{X}}{\partial s} \times \frac{\partial \mathbf{X}}{\partial t} = \mathbf{T}_s \times \mathbf{T}_t$ is nonzero.

Computing, we find

$$\begin{aligned} \mathbf{T}_s &= (\cos t, \sin t, 2s) \\ \mathbf{T}_t &= (-s\sin t, s\cos t, 0) \\ \mathbf{T}_s \times \mathbf{T}_t &= -2s^2\cos t\,\mathbf{i} - 2s^2\sin t\,\mathbf{j} + s\mathbf{k}. \end{aligned}$$

Clearly, $\mathbf{T}_s \times \mathbf{T}_t = 0$ whenever $s = 0$ and nonzero otherwise. When $s = 0$, we have $\mathbf{X}(0,t) = (0,0,0)$. Therefore, S is smooth everywhere but the origin.

The preimage of $(1, \sqrt{3}, 4)$ is $(2, \frac{\pi}{3})$. By above, the normal to S at $(1, \sqrt{3}, 4)$ is $\mathbf{T}_s \times \mathbf{T}_t(2, \frac{\pi}{3}) = (-2 \cdot 2^2 \cdot \frac{1}{2}, -2 \cdot 2^2 \cdot \frac{\sqrt{3}}{2}, 2) = (-4, -4\sqrt{3}, 2) = -2 \cdot (2, 2\sqrt{3}, -1)$. Therefore, the tangent plane at our point is given by the equation

$$2(x-1) + 2\sqrt{3}(y - \sqrt{3}) - (z - 4) = 0.$$

(b) We have $x^2 + y^2 = s^2\cos^2 t + s^2\sin^2 t = s^2(\cos^2 t + \sin^2 t) = s^2 = z$, which gives the desired equation (taking $f(x,y) = x^2 + y^2$).

(c) We see that $f(x,y)$ is everywhere infinitely differentiable. Regarding S now as the graph of $f(x,y)$, we then have that S has a tangent plane at every point. Let's find its equation. We have

$$f_x(x,y) = 2x \quad \text{and} \quad f_y(x,y) = 2y,$$

So that the equation of a tangent plane to S at the point (a,b) is given by

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) = a^2 + b^2 + 2a(x-a) + 2b(y-b).$$

In particular, taking $(a,b) = (0,0)$, the tangent plane to S at $(0,0,0)$ is defined by the equation

$$z = 0.$$

Remark. Taking $(a,b) = (1, \sqrt{3})$, the tangent plane to S at $(1, \sqrt{3}, 4)$ is defined by the equation

$$z = 1 + 3 + 2(x-1) + 2\sqrt{3}(y - \sqrt{3}).$$

We see that this is the same equation as part (a), so that our two methods give consistent answers when they both apply.

Remark. This exercise gives an example of a surface S with a 'non-smooth point' p where a tangent plane to S at p nevertheless exists.

2 (7.1.9, p. 418)**.** *Verify that for the standard torus T (cf. p. 408) the s-coordinate curve, when $t = t_0$, is a circle of radius $a + b\cos t_0$.*

Solution. The torus T is parametrized by

$$\begin{cases} x = (a + b\cos t)\cos s \\ y = (a + b\cos t)\sin s \\ z = b\sin t \end{cases} \quad \begin{array}{l} 0 \le s,\ t \le 2\pi; \\ a, b \text{ positive constants with } a > b \end{array}$$

Therefore, the s-coordinate curve for $t = t_0$ is

$$s \mapsto ((a + b\cos t_0)\cos s, (a + b\sin t_0)\sin s, b\sin t_0).$$

Note that the z-coordinate is constant for all s. Thus, the trace of the curve is contained in a vertical translate of the xy-plane. Now, computing the sum of squares of the first two coordinates, we have

$$(a + b\cos t_0)^2\cos^2 s + (a + b\cos t_0)^2\sin^2 s = (a + b\cos t_0)^2.$$

This is the equation of a circle of radius $(a + b\cos t_0)$.

3 (7.2.2, p. 438)**.** *Let $D = \{(s,t)|s^2 + t^2 \le 1,\ s \ge 0,\ t \ge 0\}$ and let $\mathbf{X} : D \to \mathbb{R}^3$ be defined by $\mathbf{X}(s,t) = (s + t, s - t, st)$.*

(a) Determine $\iint_{\mathbf{X}} f\,dS$, where $f(x,y,z) = 4$.

(b) Find the value of $\iint_{\mathbf{X}} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Solution. Before computing the two integrals, let's find the normal vectors to S. We have

$$\begin{aligned} \mathbf{T}_s &= (1, 1, t), \\ \mathbf{T}_t &= (1, -1, s), \\ \mathbf{N} = \mathbf{T}_s \times \mathbf{T}_t &= (s + t, t - s, -2) \text{ and} \\ \|\mathbf{N}(s,t)\| = \sqrt{(s+t)^2 + (t-s)^2 + 4} &= \sqrt{2s^2 + 2t^2 + 4} = \sqrt{2}\sqrt{s^2 + t^2 + 2}. \end{aligned}$$

(a) We have

$$\iint_{\mathbf{X}} f\,dS = \iint_D f(\mathbf{X}(s,t))\,\|\mathbf{N}(s,t)\|\,dsdt = \iint_D 4\sqrt{2}\sqrt{s^2 + t^2 + 2}\,dsdt.$$

Converting the integral to polar coordinates, the above becomes

$$2\sqrt{2}\int_0^{\pi/2}\int_0^1 2\sqrt{r^2 + 2}\,rdrd\theta.$$

Using the change of variable $u = r^2 + 2$, $du = 2rdr$, our integral is

$$2\sqrt{2}\int_0^{\pi/2}\int_2^3 \sqrt{u}\,dud\theta = 2\sqrt{2} \cdot \frac{\pi}{2} \cdot \left[\frac{2}{3}u^{3/2}\right]_2^3 = \sqrt{2}\pi(2\sqrt{3} - \frac{4}{3}\sqrt{2}).$$

(b) We have

$$\begin{aligned}\iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{N}(s,t)\, dsdt &= \iint_D (s+t, s-t, st) \cdot (s+t, t-s, -2)\, dsdt \\ &= \iint_D (s+t)^2 - (s-t)^2 - 2st\, dsdt \\ &= \iint_D (s^2+t^2+2st) - (s^2+t^2-2st) - 2st\, dsdt \\ &= \iint_D 2st\, dsdt\end{aligned}$$

Converting to polar coordinates, we have

$$\begin{aligned}\int_0^1 \int_0^{\pi/2} 2(r\cos\theta)(r\sin\theta)\, r dr d\theta &= \left(\int_0^1 r^3\, dr\right)\left(\int_0^{\pi/2} 2\cos\theta\sin\theta\, d\theta\right) \\ &= \frac{1}{4}\int_0^{\pi/2} \sin 2\theta\, d\theta = \frac{1}{4}\left[-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2} = \frac{1}{4}.\end{aligned}$$

4 (7.2.3, p.438)**.** *Find the flux of* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *across the surface* S *consisting of the triangular region* S *of the plane* $2x - 2y + z = 2$ *that is cut out by the coordinate planes. Use an upward-pointing normal to orient* S*.*

Solution. We can read the normal to the plane $2x-2y+z = 2$ from its equation: we have $\mathbf{N} = (2, -2, 1)$ as the upward-pointing normal and hence $\mathbf{n} = \frac{1}{3}\mathbf{N}$ as the upward-pointing *unit* normal. Our next task is to parametrize the surface S. One way to do this is as follows.

First, we find the three vertices of S. Intersecting S with the plane $z = 0$ gives the line $2x - 2y = 2$, with x-intercept $(1, 0, 0)$ and y-intercept $(0, -1, 0)$. Intersecting S with the plane $x = 0$ gives the line $-2y + z = 2$, with z-intercept $(0, 0, 2)$. Let $\mathbf{v_1} = (0, 0, 2) - (0, -1, 0) = (0, 1, 2)$ and $\mathbf{v_2} = (1, 0, 0) - (0, -1, 0) = (1, 1, 0)$. Then S may be parametrized as

$$\mathbf{X}(s,t) = (0, -1, 0) + s\mathbf{v_1} + t\mathbf{v_2} = (t, s+t-1, 2s), \quad 0 \le s+t \le 1,$$

where the bounds on $s + t$ follow from the requirements that the x, y and z-coordinates of $\mathbf{X}(s,t)$ satisfy $x \ge 0$, $y \le 0$ and $z \ge 0$.

We can check that we get the same normal vector this way: $\mathbf{T}_s = (0, 1, 2) = \mathbf{v_1}$, $\mathbf{T}_t = (1, 1, 0) = \mathbf{v_2}$ and $\mathbf{N}(s,t) = \mathbf{T}_s \times \mathbf{T}_t = (-2, 2, -1)$, so that taking the upward normal gives the same vector $\mathbf{N}$ as before.

Let $D := \{(s,t) : 0 \le s+t \le 1\}$. Then the flux of $\mathbf{F}$ through S is

$$\begin{aligned}\iint_D \mathbf{F}(\mathbf{X}(s,t)) \cdot \mathbf{n}\, dsdt &= \iint_D (t, s+t-1, 2s) \cdot \frac{1}{3}(2, -2, 1)\, dsdt \\ &= \int_0^1 \int_0^{1-t} \frac{1}{3}(2t - 2s - 2t + 2 + 2s)\, dsdt \\ &= \int_0^1 \int_0^{1-t} \frac{2}{3}\, dsdt = \int_0^1 \frac{2}{3}(1-t)dt = \frac{2}{3}(1 - \frac{1}{2}) = \frac{1}{3}.\end{aligned}$$