Exercise 1 (2.2.8 pg. 107)**.** *Evaluate*

$$\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$$

or explain why it does not exist.

Solution. Along the line $y = 0$, our function is equal to the constant function

$$\frac{0}{\sqrt{x^2+0^2}} = 0,$$

(except at $(0,0)$, where it is undefined). We then have

$$\lim_{(x,y)\to(0,0) \text{ along } y=0} \frac{|y|}{\sqrt{x^2+y^2}} = 0.$$

On the other hand, along the line $x = 0$, our function is equal to the constant function

$$\frac{|y|}{\sqrt{0^2+y^2}} = \frac{|y|}{|y|} = 1,$$

(once again, except at $(0,0)$, where it is undefined). Hence, we have

$$\lim_{(x,y)\to(0,0) \text{ along } x=0} \frac{|y|}{\sqrt{x^2+y^2}} = 1 \neq 0.$$

But if it exists, $\lim_{(x,y)\to(0,0)} \frac{|y|}{\sqrt{x^2+y^2}}$ doesn't depend on the choice of path along which (x,y) approaches $(0,0)$. So the limit doesn't exist.

Exercise 2 (2.2.23, pg. 107)**.** *Examine the behaviour of* $f(x,y) = x^4y^4/(x^2+y^4)^3$ *as* (x,y) *approaches* $(0,0)$ *along various straight lines. From your observations, what might you conjecture* $\lim_{(x,y)\to(0,0)} f(x,y)$ *to be? Next, consider what happens when* (x,y) *approaches* $(0,0)$ *along the curve* $x = y^2$*. Does* $\lim_{(x,y)\to(0,0)}$ *exist? Why or why not?*

Solution. Along the line $y = mx$, our function is equal to

$$f(x,mx) = \frac{x^4(m^4x^4)}{(x^2+m^4x^4)^3} = \frac{m^4x^8}{x^6(1+m^4x^2)^3} = \frac{m^4x^2}{(1+m^4x^2)^3}.$$

So we have

$$\lim_{(x,y)\to(0,0) \text{ along } y=mx} f(x,y) = \lim_{x\to 0} \frac{m^4x^2}{(1+m^4x^2)^3} = 0.$$

Similarly, we can check that the limit of $f(x,y)$ as (x,y) goes to $(0,0)$ along the line $x = 0$ is 0. In particular, the limit of $f(x,y)$ as (x,y) goes to $(0,0)$ along any straight line exists and does not depend on the choice of line.

However, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist! To see this, we consider the values $f(x,y)$ takes along the parabola $x = y^2$. We have

$$f(y^2,y) = \frac{y^8y^4}{(y^4+y^4)^3} = \frac{y^{12}}{8y^{12}} = \frac{1}{8}.$$

Hence,

$$\lim_{(x,y)\to(0,0) \text{ along } x=y^2} f(x,y) = \frac{1}{8} \neq 0$$

which shows that the limit of $f(x,y)$ as (x,y) goes to $(0,0)$ doesn't exist.

Exercise 3 (2.2.40, pg. 108)**.** *Determine whether the following function is continuous throughout its domain:*

$$g(x,y) = \begin{cases} \frac{x^3+x^2+xy^2+y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 2, & \text{if } (x,y) = (0,0). \end{cases} .$$

Solution. The domain of g is all of $\mathbb{R}^2$. As $x^3 + x^2 + xy^2 + y^2$ and $x^2 + y^2$ are polynomials, they are continuous on all of $\mathbb{R}^2$. Therefore, their quotient $g(x,y)$ is continuous where $x^2 + y^2$ doesn't vanish, that is, at all points of $\mathbb{R}^2$ except possibly the origin. We check whether $g(x,y)$ is continuous at the origin.

Rearranging,

$$x^3 + x^2 + xy^2 + y^2 = x(x^2 + y^2) + x^2 + y^2 = (x+1)(x^2 + y^2).$$

So for $(x,y) \neq (0,0)$,

$$g(x,y) = \frac{x^3 + x^2 + xy^2 + y^2}{x^2 + y^2} = \frac{(x+1)(x^2 + y^2)}{(x^2 + y^2)} = (x+1).$$

Thus, the limit of $g(x,y)$ as (x,y) goes to $(0,0)$ exists and is equal to

$$\lim_{(x,y)\to(0,0)} (x+1) = 1.$$

On the other hand, $g(0,0) = 2$ by definition, so $g(x,y)$ is not continuous at the origin.