Exercise 1 (2.5.22, pg. 269)**.** *Find the point on the surface $x^3 - 2y^2 + z^2 = 27$ where the tangent plane is perpendicular to the line given parametrically as $x = 3t-5$, $y = 2t+7$, $z = 1-\sqrt{2}t$.*

Solution. First, we want to find the normal vector to the tangent plane at every point $\mathbf{x_0}$ on our surface. By theorem 6.4 on page 159, this is just $\nabla f(\mathbf{x_0})$. Computing,

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) = \left(3x^2, -4y, 2z\right).$$

Now the points on our line are given by

$$\left(-5, 7, 1\right) + t\left(3, 2, -\sqrt{2}\right).$$

In particular, the direction vector of the line is $\left(3, 2, -\sqrt{2}\right)$. We observe that *the line is perpendicular to the plane exactly when the direction vector of the line is parallel to the normal vector of the plane.* Expressing this in vector language, we want to find all points $\left(x, y, z\right)$ on the surface such that

$$\nabla f(x, y, z) = \left(3x^2, -4y, 2z\right) = \lambda\left(3, 2, -\sqrt{2}\right)$$

for some nonzero real number λ. Moreover, since $\left(x, y, z\right)$ must lie on the surface, we have the necessary condition

$$x^3 - 2y^2 + z^2 = 27. \tag{1}$$

Now

$$\begin{cases} 3x^2 = 3\lambda \\ -4y = 2\lambda \\ 2z = -\sqrt{2}\lambda \end{cases} \quad \text{implies} \quad \begin{cases} x^2 = \lambda \\ y = -\frac{\lambda}{2} \\ z = -\frac{\lambda}{\sqrt{2}} \end{cases}.$$

The two possible solutions of $x^2 = \lambda$ are $x = \sqrt{\lambda}$ and $x = -\sqrt{\lambda}$. We show that in the first case we can find exactly one point on the surface at which the tangent plane is perpendicular to our line and in the second case no such points.

Suppose first that $x = \sqrt{\lambda}$. Plugging into equation (1) gives

$$27 = \lambda^{3/2} - \frac{2\lambda^2}{4} + \frac{\lambda^2}{2} = \lambda^{3/2},$$

which has the unique solution $\lambda = 9$. The corresponding point on our surface is $\left(3, -\frac{9}{2}, -\frac{9}{\sqrt{2}}\right)$.

On the other hand, if we require $x = -\sqrt{\lambda}$, plugging into equation (1) gives the condition $27 = -\lambda^{3/2}$, which has no solutions as $\lambda^{3/2} \geq 0$, as desired.

Exercise 2 (2.6.27, pg. 169)**.** *Consider the surface S defined by the equation $f(x, y, z) = x^3 - x^2y^2 + z^2 = 0$.*

(a) Find an equation for the plane tangent to S at the point $\left(2, -3/2, 1.\right)$

(b) Does S have a tangent plane at the origin?

Solution. (a) If the tangent plane at point $\mathbf{x_0} \in S$ exists, is equation is given by

$$\nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) = 0.$$

Computing,

$$\nabla f(x, y, z) = \left(3x^2 - 2xy^2, -2x^2y, 2z\right)$$

Then the equation of the tangent plane at $\big(x_0, y_0, z_0\big)$ is

$$0 = \nabla f\,\big(x_0, y_0, z_0\big)\cdot\big(x - x_0, y - y_0, z - z_0\big) = (3x_0^2 - 2x_0y_0^2)(x - x_0) - 2x_0^2y_0(y - y_0) + 2z_0(z - z_0).$$

In particular, at $(x_0, y_0, z_0) = (2, -3/2, 1)$, this becomes

$$0 = (12 - 9)(x - 2) + 12(y + 3/2) + 2(z - 1) = 3x + 12y + 2z + 10.$$

(b) Since $\nabla f(0, 0, 0) = \big(0, 0, 0\big)$, the surface doesn't have a tangent plane at the origin.

Exercise 3 (2.6.34, pg. 169)**.** *Consider the surface defined by the equation*

$$f(x, y, z) := x^3z + x^2y^2 + \sin(yz) = -3.$$

(a) Find an equation for the plane tangent to S at the point $(-1, 0, 3)$.

(b) The ***normal line*** *to a surface S in $\mathbb{R}^3$ at a point $\big(x_0, y_0, z_0\big)$ on it is the line that passes through $\big(x_0, y_0, z_0\big)$ and is perpendicular to S. Find a set of parametric equations for the line normal to the surface given above at the point $(-1, 0, 3)$.*

Solution. (a) The gradient of f is

$$\nabla f(x, y, z) = \big(3x^2z + 2xy^2, 2x^2y + z\cos(yz), x^3 + y\cos(yz)\big)\,.$$

Therefore, the equation of the plane tangent to the surface at $\big(x_0, y_0, z_0\big)$ is

$$0 = (3x_0^2z_0 + 2x_0y_0^2)(x - x_0) + (2x_0^2y_0 + z_0\cos(y_0z_0))(y - y_0) + (x_0^3 + y_0\cos(y_0z_0))(z - z_0).$$

Then at the point $(-1, 0, 3)$, $\nabla f(-1, 0, 3) = \big(3 \cdot 1 \cdot 3 + 0, 0 + 3 \cdot 1, -1 + 0\big) = \big(9, 3, -1\big)$ and the above becomes

$$0 = 9(x + 1) + 3y - (z - 3) = 9x + 3y - z + 12.$$

(b) By definition, the direction vector of the normal line is the normal vector to the surface S at $\big(x_0, y_0, z_0\big)$. But we know that this vector is nothing but $\nabla f\,\big(x_0, y_0, z_0\big)$. Thus the points of a general normal line may be parametrized as

$$\big(x_0, y_0, z_0\big) + t\,\nabla f\,\big(x_0, y_0, z_0\big)\,.$$

In our case, then, the normal line to S at $(-1, 0, 3)$ is given parametrically by

$$(-1, 0, 3) + t(9, 3, -1),$$

using the values of the partials of f evaluated at $(-1, 0, 3)$ in part (a). Equivalently, the desired set of parametric equations is $x = 9t - 1$, $y = 3t$, $z = 3 - t$.