Exercise 1 (3.3.22, pg. 213). *Calculate the flow line $\mathbf{x}(t)$ of the vector field $\mathbf{F}(x, y, z) = 2\mathbf{i} - 3y\mathbf{j} + z^3\mathbf{k}$ with $\mathbf{x}(0) = (3, 5, 7)$.*

Solution. Writing $\mathbf{x}(t) = (x(t), y(t), z(t))$ and comparing coordinates, we get the system of equations

$$\begin{cases} \frac{dx}{dt} = 2 \\ \frac{dy}{dt} = -3y \\ \frac{dz}{dt} = z^3 \end{cases}$$

with initial conditions $x(0) = 3, y(0) = 5$ and $z(0) = 7$.

Since the time-derivative with respect to one of the variables doesn't involve the other two (for example, $y'(t)$ is independent of $x(t)$ and $z(t)$ and similarly for $x'(t)$ and $z'(t)$), our problem amounts to solving three separate ODEs. Let's tackle them in turn.

First, $x'(t) = 2$ is immediately solved by integration of both sides to get $x(t) = 2t + C$. Using the initial condition $x(0) = 3$, we get $C = 3$ and

$$x(t) = 2t + 3.$$

Second, separate variables in $\frac{dy}{dt} = -3y$ to get $\frac{dy}{y} = -3\,dt$. Integrating both sides, we have $\ln(y(t)) = -3t + C$. Using the initial condition $y(0) = 5$, we have $\ln(5) = C$. Finally, taking exponential of both sides,

$$y(t) = 5\,e^{-3t}.$$

Third, separate variables in $\frac{dz}{dt} = z^3$ to get $\frac{dz}{z^3} = 1\,dt$. Integrating both sides, we have $-\frac{1}{2}\frac{1}{z^2} = t + C$. Using the initial condition $z(0) = 7$, get $-\frac{1}{98} = C$. So $\frac{1}{z(t)^2} = -2t + \frac{1}{49} = \frac{1-98t}{49}$. Therefore,

$$z(t) = \frac{7}{\sqrt{1 - 98t}}.$$

Summarizing, our flow line is given for all t by

$$\mathbf{x}(t) = \left(2t + 3,\ 5\,e^{-3t},\ \frac{7}{\sqrt{1 - 98t}}\right).$$

Exercise 2 (3.3.24, pg. 213). *Consider the vector field $\mathbf{F} = 2x\,\mathbf{i} + 2y\,\mathbf{j} - 3\,\mathbf{k}$.*

(a) Show that $\mathbf{F}$ is a gradient field.

(b) Describe the equipotential surfaces of $\mathbf{F}$.

Solution. (a) We would like to find a scalar-valued function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla f = \mathbf{F}$ at all points of $\mathbb{R}^3$. We see that $f(x, y, z) = x^2 + y^2 - 3z$ works.

(b) The equipotential surfaces of $\mathbf{F}$ are given by $f(x, y, z) = c$ for some real number c. Using the function found in (a), we have $x^2 + y^2 - 3z = c$ or $x^2 + y^2 = 3z + c$, which is a vertical translation of a paraboloid (the surface obtained by rotating a parabola in the xy-plane around the z-axis).

Exercise 3 (3.4.17, 18, pg. 222). *Let $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and let r denote $\|\mathbf{r}\|$. Verify that*

(a) $\nabla r^n = n r^{n-2}\mathbf{r}$

(b) $\nabla(\ln r) = \frac{\mathbf{r}}{r^2}$

Solution. (a) Note that $r^n = (x^2+y^2+z^2)^{n/2}$. Then, by the chain rule,

$$\frac{\partial r^n}{\partial x} = \frac{n}{2}\left(x^2+y^2+z^2\right)^{(n/2)-1}\,2x = nx\,\left(x^2+y^2+z^2\right)^{(n-2)/2} = nx\,r^{n-2}$$

and similarly $\frac{\partial r^n}{\partial y} = ny\,r^{n-2}$ and $\frac{\partial r^n}{\partial z} = nz\,r^{n-2}$. Then

$$\nabla r^n = \frac{\partial r^n}{\partial x}\,\mathbf{i} + \frac{\partial r^n}{\partial y}\,\mathbf{j} + \frac{\partial r^n}{\partial z}\,\mathbf{k} = nr^{n-2}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) = nr^{n-2}\,\mathbf{r}.$$

(b) We have

$$\frac{\partial \ln r}{\partial x} = \frac{\frac{\partial}{\partial x}\left((x^2+y^2+z^2)^{1/2}\right)}{(x^2+y^2+z^2)^{1/2}} = \frac{\frac{1}{2}\,(x^2+y^2+z^2)^{-1/2}\,2x}{(x^2+y^2+z^2)^{1/2}} = \frac{x}{(x^2+y^2+z^2)} = \frac{x}{r^2}$$

and similarly $\frac{\partial \ln r}{\partial y} = \frac{y}{r^2}$ and $\frac{\partial \ln r}{\partial z} = \frac{z}{r^2}$. Then

$$\nabla(\ln r) = \frac{\partial r^n}{\partial x}\,\mathbf{i} + \frac{\partial r^n}{\partial y}\,\mathbf{j} + \frac{\partial r^n}{\partial z}\,\mathbf{k} = \frac{x}{r^2}\,\mathbf{i} + \frac{y}{r^2}\,\mathbf{j} + \frac{z}{r^2}\,\mathbf{k} = \frac{\mathbf{r}}{r^2}.$$