Exercise 1 (5.2.27, pg. 308). *Find the volume of the region under the graph of*

$$f(x,y) = 2 - |x| - |y|$$

and above the xy-plane.

Solution. The graph of $f(x,y)$ will be above the xy-plane exactly when $f(x,y) \geq 0$; therefore, we begin by finding the region $\{(x,y) \in \mathbb{R}^2 : f(x,y) \geq 0\}$. It is convenient to split the domain of f into the following four regions: $x \geq 0, y \geq 0$; $x \geq 0, y \leq 0$; $x \leq 0, y \geq 0$ and $x \leq 0, y \leq 0$. On these regions, $f(x,y)$ is equal to

$$f(x,y) = \begin{cases} 2 - x - y, & x \geq 0,\ y \geq 0 \\ 2 - x + y, & x \geq 0,\ y \leq 0 \\ 2 + x - y, & x \leq 0,\ y \geq 0 \\ 2 + x + y, & x \leq 0,\ y \leq 0 \end{cases}.$$

Let's consider the region $x \geq 0$, $y \geq 0$ and find the intersection

$$\{(x,y) : f(x,y) \geq 0\} \cap \{(x,y) : x \geq 0, y \geq 0\}.$$

On this region, $f(x,y) = 2 - x - y$. Therefore, $f(x,y) \geq 0$ exactly when $x + y \leq 2$ or, equivalently, $y \leq 2 - x$. The inequalities $x \geq 0$, $y \geq 0$ and $y \leq 2 - x$ describe a triangle with vertices $(0,0)$, $(2,0)$ and $(0,2)$.

Similarly, we find that the intersection of $\{(x,y) : f(x,y) \geq 0\}$ with the other three regions are triangles (the easy way to see this is to observe that $f(x,y)$ is *symmetric with respect to exchanging x and y by $-x$ and $-y$, as well as interchanging x and y*, but we can also find the triangles by the same method as above). Explicitly, the triangles are

$x \geq 0$, $y \leq 0$**:** $(0,0)$, $(2,0)$ and $(0,-2)$.

$x \leq 0$, $y \geq 0$**:** $(0,0)$, $(-2,0)$ and $(0,2)$.

$x \leq 0$, $y \leq 0$**:** $(0,0)$, $(-2,0)$ and $(0,-2)$.

Together, the four triangles form a regular rhombus with vertices $(0,2)$, $(2,0)$, $(0,-2)$ and $(-2,0)$. Let's call this rhombus R. Summarizing, we found that $\{(x,y) : f(x,y) \geq 0\} = R$.

By definition, the volume under the graph of $f(x,y)$ and above the xy-plane is given by the double integral

$$\int\int_R f(x,y)\, dA.$$

We now want to show that R is a region of 'Type 3' and use our theorem 2.10 to evaluate the integral (cf. pp. 299–300 of the textbook). Indeed, we can describe R as

$$R = \{(x,y) : -2 + |x| \leq y \leq 2 - |x|,\ -2 \leq x \leq 2\},$$

so R is Type 1 and

$$R = \{(x,y) : -2 \leq y \leq 2,\ -2 + |y| \leq x \leq 2 - |y|\},$$

so R is Type 2. By Theorem 2.10, we can use either parametrization of R to evaluate the double integral. Let's arbitrarily choose to use the first. Then

$$\int\int_R f(x,y)\, dA = \int_{-2}^{2} \int_{-2+|x|}^{2-|x|} f(x,y)\, dydx = \int_{-2}^{0} \int_{-2-x}^{2+x} f(x,y)\, dydx + \int_{0}^{2} \int_{-2+x}^{2-x} f(x,y)\, dydx.$$

Let's evaluate the two integrals on the right hand side in turn. For the first one, we have

$$\int_{-2}^{0}\int_{-2-x}^{2+x} 2-|x|-|y|\,dydx = \int_{-2}^{0}\int_{-2-x}^{0} 2+x+y\,dydx + \int_{-2}^{0}\int_{0}^{2+x} 2+x-y\,dydx.$$

Again, evaluating these in turn, we have for the first integral

$$\begin{aligned}\int_{-2}^{0}\int_{-2-x}^{0} 2+x+y\,dydx &= \int_{-2}^{0}\left(2(0-(-2-x))+x(0-(-2-x))+\left[\frac{y^2}{2}\right]_{-2-x}^{0}\right)dx\\ &= \int_{-2}^{0} 2(2+x)+x(2+x)+\left(0-\frac{(-2-x)^2}{2}\right)dx\\ &= \int_{-2}^{0}(2+x)(2+x)-\frac{(2+x)^2}{2}\,dx\\ &= \int_{-2}^{0}\frac{(2+x)^2}{2}\,dx.\end{aligned}$$

Changing variable of integrating to $u=2+x$, $du=dx$, the above integral becomes

$$\int_0^2 \frac{u^2}{2}\,du = \frac{1}{6}\left[u^3\right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

Similarly, we find that the other three integrals are each equal to $\frac{4}{3}$, so that the volume under the graph of $f(x,y)$ and above the xy-axis is $4\cdot\frac{4}{3}=\frac{16}{3}$.

Exercise 2 (5.3.12, pg. 312)**.** *Rewrite the following sum as a single integral by reversing the order of integration and evaluate:*

$$\int_0^1\int_0^x \sin x\,dydx + \int_1^2\int_0^{2-x}\sin x\,dydx$$

Solution. To exchange the order of integration, we have to understand the shape of the region of integration. The first integral is taken over the region

$$\{(x,y): 0\le x\le 1,\ 0\le y\le x\},$$

which is equal to the triangle with vertices $(0,0)$, $(1,1)$ and $(1,0)$. Taking vertical slices, this region may be described as

$$\{(x,y): 0\le y\le 1,\ y\le x\le 1\}.$$

Therefore, the first integral with order of integration reversed is equal to

$$\int_0^1\int_y^1 \sin x\,dxdy.$$

The region of integration for the second integral is

$$\{(x,y): 1\le x\le 2,\ 0\le y\le 2-x\}.$$

this is a triangle with vertices $(1,0)$, $(1,1)$ and $(2,0)$. Taking vertical slices instead of horizontal ones, we have the alternative description

$$\{(x,y): 0\le y\le 1,\ 1\le x\le 2-y\},$$

so that the second integral with order of integration reversed is equal to

$$\int_0^1 \int_1^{2-y} \sin x \, dxdy.$$

Then the sum is

$$\int_0^1 \int_y^1 \sin x \, dxdy + \int_0^1 \int_1^{2-y} \sin x \, dxdy = \int_0^1 \int_y^{2-y} \sin x \, dxdy.$$

Let's now evaluate the integral. We have

$$\begin{aligned} \int_0^1 \int_y^{2-y} \sin x \, dxdy &= \int_0^1 [-\cos x]_y^{2-y} \; dy \\ &= -\int_0^1 \cos(2-y) \, dy + \int_0^1 \cos y \, dy \end{aligned}$$

Using the substitution $u = 2 - y, \; du = -dy$ in the first integral, we have

$$\begin{aligned} & -\int_1^2 \cos u \, du + \int_0^1 \cos y \, dy \\ &= -\left[\sin u\right]_1^2 + \left[\sin y\right]_0^1 \\ &= -\sin(2) + \sin(1) + \sin(1) - \sin(0) \\ &= 2\sin(1) - \sin(2). \end{aligned}$$

Exercise 3 (5.3.14, pg. 312)**.** *Evaluate:*

$$\int_0^1 \int_{3y}^3 \cos(x^2) \, dxdy$$

Solution. The region of integration is

$$\{(x, y) : 0 \le y \le 1, \; 3y \le x \le 3\}.$$

This is a triangle with vertices $(0, 0)$, $(3, 0)$ and $(3, 1)$. Taking vertical slices, this region admits the alternative description

$$\{(x, y) : 0 \le x \le 3, \; 0 \le y \le \frac{x}{3}\}.$$

Therefore, our double integral with the order of integration reversed is equal to

$$\int_0^3 \int_0^{x/3} \cos(x^2) \, dydx = \int_0^3 \frac{x}{3} \cos(x^2) \, dx.$$

Making the change of variable of integration $u = x^2$, $du = 2x \, dx$, the above integral is equal to

$$\frac{1}{6} \int_0^9 \cos u \, du = \frac{\sin 9}{6}.$$