Exercise 1 (5.5.6, p. 341). *Suppose $\mathbf{T}(u,v) = (u, uv)$. Explain how $\mathbf{T}$ transforms the unit square $D^* = [0,1] \times [0,1]$. Is $\mathbf{T}$ one-to-one on D^*?*

Solution. Note that $\mathbf{T}$ is not linear, so we can't use the techniques described in the beginning of Chapter 5.5 to analyze $\mathbf{T}$. Instead, let's try to get an idea of what $\mathbf{T}$ looks like by analyzing where the vertical and horizontal line segments in D^* are mapped by $\mathbf{T}$.

Horizontal line segments (i.e. ones parallel to the u-axis) can be described by equations $v = c$ for fixed $0 \le c \le 1$. The image of such a line segment is the set $\{(x,y) = (u, cu) : 0 \le u \le 1\}$. This is a line segment of slope c and with endpoints $(0,0)$ and $(1,c)$. As c varies from 0 to 1, these line segments sweep out a triangle with vertices $(0,0), (1,1)$ and $(1,0)$.

We now turn to vertical line segments (i.e. ones parallel to the v-axis) can be described by equations $u = c$ for some fixed $0 \le c \le 1$. The image of such a line is the set $\{(x,y) = (c, cv) : 0 \le v \le 1\}$. This is a vertical line segment of length c with endpoints $(c,0)$ and (c,c). We can think of it as 'shrinking' the initial line segment by parameter c. Of course, as c varies from 0 to 1, these line segments sweep out the same triangle as before.

Note in particular that the line segment $u = 0$ is collapsed to the origin (in other words, every point of the form $(0,v)$ with $0 \le v \le 1$ is mapped to $(0,0)$), so $\mathbf{T}$ is not one-to-one (if we exclude the line $u = 0$ from the domain, $\mathbf{T}$ becomes one-to-one).

Exercise 2 (5.5.9, p. 342). *Evaluate the integral*

$$\int_0^2 \int_{x/2}^{(x/2)+1} x^5 (2y - x) e^{(2y-x)^2} \, dy dx$$

by making the substitution $u = x$, $v = 2y - x$.

Solution. We use the notation of Theorem 5.3 on p. 329. Our region of integration in the xy-plane is $D := \{(x,y) : 0 \le x \le 2,\ \frac{x}{2} \le y \le \frac{x}{2} + 1\}$.

First, let's solve for x and y as functions of u and v. We have $x(u,v) = u$ and $y(u,v) = \frac{u+v}{2}$. Hence, our change of variables map is $\mathbf{T}(u,v) = (u, \frac{u+v}{2})$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \frac{1}{2} - 0 = \frac{1}{2}.$$

Next, we need to find a region D^* in the uv-plane that is mapped to D by $\mathbf{T}$. For this, let's describe the boundary of D. The region D is a parallelogram with vertices at the points $(0,0)$, $(1,0)$, $(2,1)$ and $(2,2)$ of the xy-plane. The four sides of the boundary are described by the equations $x = 0$, $x = 2$, $y = \frac{x}{2}$ and $y = \frac{x}{2}+1$. We have $x = 0 \iff u = 0$ and $x = 2 \iff u = 2$. Moreover,

$$\begin{aligned} y = \frac{x}{2} &\iff \frac{u+v}{2} = \frac{u}{2} \iff v = 0 \quad \text{and} \\ y = \frac{x}{2} + 1 &\iff \frac{u+v}{2} = \frac{u}{2} + 1 \iff v = 2. \end{aligned}$$

Hence, we see that the region $D^* := \{(u,v) : 0 \le u \le 2,\ 0 \le v \le 2\}$ is mapped to D by $\mathbf{T}$. Finally, note that $2y - x = 2(\frac{u+v}{2}) - u = v$. Applying the change of variables theorem (Theorem 5.3), our integral becomes

$$\int_0^2 \int_0^2 u^5 v e^{v^2} \frac{1}{2} dv du.$$

Using the single-variable substitution $w = v^2$, $dw = 2v\,dv$, we have

$$\int_0^2 \int_0^2 u^5 v e^{v^2} \frac{1}{2} dv du = \frac{1}{4} \int_0^2 \int_0^4 u^5 e^w \, dw du = \frac{1}{4}(e^4 - 1) \int_0^2 u^5 \, du = \frac{2^6}{6} \frac{1}{4}(e^4 - 1) = \frac{8}{3}(e^4 - 1).$$

Exercise 3 (5.5.14, pg. 342)**.** *Convert to an integral in polar coordinates and evaluate:*

$$\int_0^2 \int_0^{\sqrt{4-x^2}} dy dx.$$

Solution. The polar-to-cartesian change of coordinates map is $\mathbf{T}(r, \theta) = (r\cos\theta, r\sin\theta)$. As usual, we need to find a region D^* in the $r\theta$-plane that is mapped to $D := \{(x, y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{4 - x^2}\}$. Our region's boundaries are described by $x = 0$, $y = 0$ and $y = \sqrt{4 - x^2}$. Moreover, have $x \ge 0$ and $y \ge 0$ for all $(x, y) \in D$. We have

$$\begin{aligned} x = 0 &\iff r\cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}, \\ y = 0 &\iff r\sin\theta = 0 \iff \sin\theta = 0 \iff \theta = 0, \\ y = \sqrt{4 - x^2} &\iff r^2\sin^2\theta = 4 - r^2\cos^2\theta \iff r^2 = 4 \iff r = 2. \end{aligned}$$

We then see that $D^* = \{(r, \theta) : 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$ is mapped to D by $\mathbf{T}$.

Remark. We could also have recognized that D is a quarter-circle of radius 2 in the first quadrant. If we noticed this, we could see the description $D^* = \{(r, \theta) : 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}\}$ right away and avoided the work of analyzing the boundary of D.

The Jacobian for polar-to-cartesian change of coordinates is just r (p. 331), so our integral is equal to

$$\int_0^2 \int_0^{\pi/2} r \, d\theta dr = \frac{\pi}{2} \int_0^2 r \, dr = \frac{\pi}{2} \frac{1}{2}(4 - 0) = \pi.$$

Note that a cirlce of radius 2 has area 4π, so indeed a quarter-circle of radius 2 should have area π.