1 (6.1.2, p. 379). *Calculate*

$$\int_{\mathbf{x}} f\,ds,$$

where $f(x,y,z) = xyz$ *and* $\mathbf{x}(t) = (t, 2t, 3t),\ 0 \le t \le 2.$

Solution. By definition,

$$\int_{\mathbf{x}} f\,ds = \int_0^2 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt.$$

We have

$$\begin{aligned} f(\mathbf{x}(t)) &= t(2t)(3t) = 6t^2 \quad \text{and} \\ \mathbf{x}'(t) &= (1,2,3), \quad \text{hence } \|\mathbf{x}'(t)\| = \sqrt{1+4+9} = \sqrt{13}. \end{aligned}$$

Therefore,

$$\int_{\mathbf{x}} f\,ds = \int_0^2 6t^2\sqrt{13}\,dt = 6\sqrt{13}\int_0^2 t^2\,dt = 6\sqrt{13}\,\frac{2^3-0}{3} = 16\sqrt{13}.$$

2 (6.1.5, p. 379). *Calculate*

$$\int_{\mathbf{x}} f\,ds,$$

where $f(x,y,z) = 2x - y^{1/2} + 2z^2$ *and* $\mathbf{x}(t) = \begin{cases} (t,t^2,0), & 0 \le t \le 1 \\ (1,1,t-1) & 1 \le t \le 3. \end{cases}$.

Solution. We have

$$\int_{\mathbf{x}} f\,ds = \int_0^3 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt = \int_0^1 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt + \int_1^3 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt.$$

For $0 \le t \le 1$,

$$\begin{aligned} f(\mathbf{x}(t)) &= 2t - (t^2)^{1/2} + 0 = t \quad \text{and} \\ \mathbf{x}'(t) &= (1,2t,0), \text{ hence } \|\mathbf{x}'(t)\| = \sqrt{1+4t^2}. \end{aligned}$$

Therefore,

$$\int_0^1 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt = \int_0^1 t\sqrt{1+4t^2}\,dt.$$

Changing variables to $u = 1 + 4t^2$, $du = 8t\,dt$, the last integral is equal to

$$\frac{1}{8}\int_1^5 \sqrt{u}\,du = \frac{1}{8}\left[\frac{2}{3}u^{3/2}\right]_1^5 = \frac{1}{12}(5^{3/2}-1).$$

For $1 \le t \le 3$,

$$\begin{aligned} f(\mathbf{x}(t)) &= 2 - 1 + 2(t-1)^2 = 1 + 2(t-1)^2 \quad \text{and} \\ \mathbf{x}'(t) &= (0,0,1), \text{ hence } \|\mathbf{x}'(t)\| = 1. \end{aligned}$$

Therefore,

$$\int_1^3 f(\mathbf{x}(t))\,\|\mathbf{x}'(t)\|\,dt = \int_1^3 (1 + 2(t-1)^2)\,dt = (3-1) + \int_1^3 2(t-1)^2\,dt.$$

Using the change of variables $u = t - 1$, $du = dt$, the last integral becomes

$$\int_1^3 2(t-1)^2\,dt = \int_0^2 2u^2\,du = \left[\frac{2u^3}{3}\right]_0^2 = \frac{16}{3} = \frac{64}{12}.$$

Hence,

$$\int_1^3 f(\mathbf{x}(t))\ \|\mathbf{x}'(t)\|\ dt = 2 + \frac{64}{12} = \frac{88}{12}$$

and

$$\int_{\mathbf{x}} f\,ds = \frac{1}{12}(5^{3/2} - 1) + \frac{88}{12} = \frac{5^{3/2} + 87}{12}.$$

3 (6.1.17, p. 379)**.** *Let* $\mathbf{F} = (2z^5 - 3xy)\mathbf{i} - x^2\mathbf{j} + x^2z\mathbf{k}$. *Calculate the line integral of* $\mathbf{F}$ *around the perimter of the square with vertices* $(1,1,3)$, $(-1,1,3)$, $(-1,-1,3)$ *and* $(1,-1,3)$, *oriented counterclockwise about the* z*-axis.*

Solution. Let's parametrize the sides S_1 (the one going from $(1,1,3)$ to $(-1,1,3)$), S_2, S_3 and S_4 of the square as follows:

$$\begin{array}{lll} S_1 & : \mathbf{x_1}(t) = (1-t, 1, 3), & 0 \le t \le 2 \\ S_2 & : \mathbf{x_2}(t) = (-1, 1-t, 3), & 0 \le t \le 2 \\ S_3 & : \mathbf{x_3}(t) = (-1+t, -1, 3), & 0 \le t \le 2 \\ S_4 & : \mathbf{x_4}(t) = (1, -1+t, 3), & 0 \le t \le 2 \end{array}$$

Then the line integral of $\mathbf{F}$ around the perimeter of the square is equal to

$$\int_{\mathbf{x_1}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{x_2}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{x_3}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{x_4}} \mathbf{F}\cdot d\mathbf{s}.$$

We have

i	$\mathbf{F}(\mathbf{x_i}(t))$	$\mathbf{x_i}'(t)$	$\mathbf{F}(\mathbf{x_i}(t)) \cdot \mathbf{x_i}'(t)$
1	$(2\cdot 3^5 - 3(1-t), (1-t)^2, 3(1-t)^2)$	$(-1,0,0)$	$-2\cdot 3^5 + 3(1-t)$
2	$(2\cdot 3^5 + 3(1-t), -1, 3)$	$(0,-1,0)$	1
3	$(2\cdot 3^5 + 3(t-1), -(t-1)^2, 3(t-1)^2)$	$(1,0,0)$	$2\cdot 3^5 + 3(t-1)$
4	$(2\cdot 3^5 - 3(t-1), -1, 3)$	$(0,1,0)$	-1

.

Hence,

$$\int_{\mathbf{x_1}} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathbf{x_3}} \mathbf{F}\cdot d\mathbf{s} \quad \text{and} \quad \int_{\mathbf{x_2}} \mathbf{F}\cdot d\mathbf{s} = -\int_{\mathbf{x_4}} \mathbf{F}\cdot d\mathbf{s}$$

and so the line integral of $\mathbf{F}$ around the perimiter of our square is equal to 0.

4 (6.1.22, pg. 379)**.** *Calculate* $\int_C zdx + xdy + ydz$, *where* C *is the curve obtained by intersecting the surfaces* $z = x^2$ *and* $x^2 + y^2 = 4$ *and oriented counterclockwise around the* $z-$*axis (as seen from the positive* $z-$*axis).*

Solution. The coordinates of points on C will simultaneously satisfy the equations $z = x^2$ and $x^2 + y^2 = 4$. In particular, $z + y^2 = 4$, hence $y^2 = 4 - z$. We see that we can split C into four paths C_1 (the one above the first quadrant), C_2, C_3 and C_4, parametrized as follows:

$$\begin{array}{lll} C_1: & \mathbf{x_1}(t) = (\sqrt{4-t}, \sqrt{t}, 4-t), & 0 \le t \le 4 \\ C_2: & \mathbf{x_2}(t) = (-\sqrt{t}, \sqrt{4-t}, t), & 0 \le t \le 4 \\ C_3: & \mathbf{x_3}(t) = (-\sqrt{4-t}, -\sqrt{t}, 4-t), & 0 \le t \le 4 \\ C_4: & \mathbf{x_4}(t) = (\sqrt{t}, -\sqrt{4-t}, t), & 0 \le t \le 4 \end{array}$$

Write $\mathbf{F}(x, y, z) = (z, x, y)$. We have

i	$\mathbf{F}(\mathbf{x_i}(t))$	$\mathbf{x_i}'(t)$	$\mathbf{F}(\mathbf{x_i}(t)) \cdot \mathbf{x_i}'(t)$
1	$(4-t, \sqrt{4-t}, \sqrt{t})$	$(-\frac{1}{2}\frac{1}{\sqrt{4-t}}, \frac{1}{2}\frac{1}{\sqrt{t}}, -1)$	$-\frac{1}{2}\frac{4-t}{\sqrt{4-t}} + \frac{1}{2}\frac{\sqrt{4-t}}{\sqrt{t}} - \sqrt{t}$
2	$(t, -\sqrt{t}, \sqrt{4-t})$	$(-\frac{1}{2}\frac{1}{\sqrt{t}}, -\frac{1}{2}\frac{1}{\sqrt{4-t}}, 1)$	$-\frac{1}{2}\frac{t}{\sqrt{t}} + \frac{1}{2}\frac{\sqrt{t}}{\sqrt{4-t}} + \sqrt{4-t}$
3	$(4-t, -\sqrt{4-t}, -\sqrt{t})$	$(\frac{1}{2}\frac{1}{\sqrt{4-t}}, -\frac{1}{2}\frac{1}{\sqrt{t}}, -1)$	$\frac{1}{2}\frac{4-t}{\sqrt{4-t}} + \frac{1}{2}\frac{\sqrt{4-t}}{\sqrt{t}} + \sqrt{t}$
4	$(t, \sqrt{t}, -\sqrt{4-t})$	$(\frac{1}{2}\frac{1}{\sqrt{t}}, \frac{1}{2}\frac{1}{\sqrt{4-t}}, 1)$	$\frac{1}{2}\frac{t}{\sqrt{t}} + \frac{1}{2}\frac{\sqrt{t}}{\sqrt{4-t}} - \sqrt{4-t}$

Adding up the integrands over the four components, we see that most cancel, except for two terms both equal to $\frac{1}{2}\frac{\sqrt{t}}{\sqrt{4-t}}$ and two terms both equal to $\frac{1}{2}\frac{\sqrt{4-t}}{\sqrt{t}}$. Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^4 \frac{\sqrt{t}}{\sqrt{4-t}} + \frac{\sqrt{4-t}}{\sqrt{t}}\, dt = \int_0^4 \frac{4}{\sqrt{4t-t^2}}\, dt$$

completing the square, we have $4t - t^2 = 4 - (t^2 - 4t + 4) = 4 - (t-2)^2$. Using the change of variables $u = t - 2$, $du = dt$, the last integral above is equal to

$$4\int_{-2}^{2} \frac{du}{\sqrt{4-u^2}}.$$

Finally, using the trigonometric substitution $u = 2\sin\theta$, $du = 2\cos\theta\, d\theta$, the last integral is equal to

$$4\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{\sqrt{4 - 4\sin^2\theta}} = 4\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta\, d\theta}{\sqrt{4\cos^2\theta}} = 4\int_{-\pi/2}^{\pi/2} d\theta = 4\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = 4\pi.$$

Alternative Solution. We begin by finding a parametrization the curve C. We know that we can parametrize the points of the cylinder $x^2 + y^2 = 4$ as $(2\cos\theta, 2\sin\theta, z),\ 0 \le \theta \le 2\pi,\ z \in \mathbb{R}$. Since the points of the curve C also satisfy $z = x^2$, C may be parametrized as $\mathbf{x}(\theta) = (2\cos\theta, 2\sin\theta, 4\cos^2\theta),\ 0 \le \theta \le 2\pi$. The tangent vectors are

$$\mathbf{x}'(\theta) = (-2\sin\theta, 2\cos\theta, -8\cos\theta\sin\theta),\ 0 \le \theta \le 2\pi.$$

Writing $\mathbf{F}(x, y, z) = (z, x, y)$, we have

$$\mathbf{F}(\mathbf{x}(\theta)) = (4\cos^2\theta, 2\cos\theta, 2\sin\theta) \text{ and}$$
$$\mathbf{F}(\mathbf{x}(\theta)) \cdot \mathbf{x}'(\theta) = -8\cos^2\theta\sin\theta + 4\cos^2\theta - 16\cos\theta\sin^2\theta.$$

Therefore,

$$\int_C \mathbf{F} \cdot d\mathbf{s} = \int_0^{2\pi} -8\cos^2\theta\sin\theta + 4\cos^2\theta - 16\cos\theta\sin^2\theta\, d\theta.$$

Now, we claim that the following equalities hold.

$$\int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta = 0.$$

To begin, we have $\cos(\theta + \pi) = -\cos(\theta)$ and $\sin(\theta + \pi) = -\sin(\theta)$ for all $\theta \in [0, 2\pi]$. Therefore, $\cos^2(\theta + \pi) = \cos^2\theta$ and $\sin^2(\theta + \pi) = \sin^2\theta$. We now show the first equality. Breaking up the integral,

$$\int_0^{2\pi} \cos^2\theta\sin\theta\, d\theta = \int_0^{\pi} \cos^2\theta\sin\theta\, d\theta + \int_{\pi}^{2\pi} \cos^2\theta\sin\theta\, d\theta,$$

and making the change of variables $u = \theta - \pi$ in the second integral, we have

$$\int_0^\pi \cos^2\theta \sin\theta \, d\theta + \int_0^\pi \cos^2(u+\pi)\sin(u+\pi)\, du = \int_0^\pi \cos^2\theta \sin\theta \, d\theta - \int_0^\pi \cos^2 u \sin u \, du = 0.$$

The other equality is shown similarly. Therefore,

$$\int_0^{2\pi} -8\cos^2\theta \sin\theta + 4\cos^2\theta - 16\cos\theta\sin^2\theta \, d\theta. = \int_0^{2\pi} 4\cos^2\theta \, d\theta.$$

Now, using the identity $\cos^2\theta = (\cos 2\theta + 1)/2$, the last integral becomes

$$\int_0^{2\pi} 2(\cos 2\theta + 1)\, d\theta = 2\int_0^{2\pi} \cos 2\theta \, d\theta + 2\int_0^{2\pi} d\theta = 0 + 4\pi = 4\pi.$$