Name and Student Number

Math/MTHE 280, Advanced Calculus, Fall 2014
Queen's University, Department of Mathematics
Please write your student number and your name clearly at the top of this page.

Additional space for calculations can be arranged on the back of each page or on additional blank pages at the end of the exam.
Do all five questions, marks are indicated. Total marks are 54.

1a) . [5 marks] Is the function f(x,y) continuous for all values of $(x, y) \in \mathbb{R}^2$? Explain your answer.

$$f(x,y) = \begin{cases} \frac{(x+y)^2}{\sqrt{x^2+y^4}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

The function is continuous at every point where the denominator is nonzero, due to the standard algebraic properties of continuous functions. This leaves only the point (0,0) to investiate. This can be done by directly examing the limiting case $\lim_{(x,y)\to(0,0)} f(x,y)$ which can be done in different ways. If polar coordinates are considered then we find that

$$\begin{aligned} \lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{(x,y)\to(0,0)} \frac{r^2(\cos(\theta)+\sin(\theta))^2}{r\sqrt{\cos(\theta)^2 + r^2\sin(\theta)^4}} \\ &= \lim_{r\to 0} r\frac{(\cos(\theta)+\sin(\theta))^2}{\sqrt{\cos(\theta)^2 + r^2\sin(\theta)^4}} \end{aligned}$$

The problem is that the fraction might not have a limit when $\cos(\theta) = 0$, and indeed it is a simple matter to check that

$$\begin{aligned} \lim_{(x,0)\to(0,0)} f(x,0) &= \lim_{(x,0)\to(0,0)} \frac{x^2}{|x|} \\ &= 0 \\ \lim_{(0,y)\to(0,0)} f(0,y) &= \frac{y^2}{y^2} \\ &= 1 \end{aligned}$$

Since the limit at (0,0) doesnt exist, the function is not continuous at (0,0).

1b) . [5 marks] Is the function $f(x, y)$ from part a) differentiable at (0,0)?. Explain your answer.

Since the function is not continuous at (0,0), it cannot be differentiable there. This can also be seen directly by examining the difference quotients for the partial derivatives

$$\lim_{(h,0)\to(0,0)} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{h^2}{h|h|} = \text{undefined}$$

$$\lim_{(0,h)\to(0,0)} \frac{f(0,h)-f(0,0)}{h} = \frac{h^2}{h^3} = \text{undefined}$$

2. For the vector field $\vec{\mathbf{F}}(x,y) = y\vec{\mathbf{i}} + x\vec{\mathbf{j}}$

a)[**4 marks**] Sketch the vector field $\vec{F}$ along the parabola $y = 4 - x^2$, between the points (3,-5) to (-2,0). Show that the gradient of the function f(x,y)=xy matches the vector field $\vec{\mathbf{F}}$.

The parabola opens downward, and has vertex at x=0, y=4. The intercepts with the x-axis occur at $x = \pm 2, y = 0$. The vector field $\vec{\mathbf{F}}(x,y) = y\vec{\mathbf{i}} + x\vec{\mathbf{j}}$ may be sketched at the endpoints F(3,-5)= -5i + 3j, F(-2,0)=-2j and at the intercepts with the x and y axes, F(2,0) = 2j, F(0,4) = 4i.
The gradient field of the function f(x,y) = xy, is $\nabla f(x,y) = yi + xj = \vec{F}(x,y)$.

b) [**6marks**] Calculate the tangent vector **v** to the parabolic curve $y = 4 - x^2$ (in the direction of increasing values of x), and calculate the directional derivative of f(x,y)=xy along the parabolic arc, in the direction of the tangent vector to the parabolic curve, between the points (3,-5) and (-2,0). Find the interval of x-values for which $D_{\mathbf{v}} f(x,y) > 0$, for the tangent vector **v**.

The tangent vector may be found in several ways. The easiest is to parameterize the parabola, $x = t, y = 4 - t^2$ then $\mathbf{v} = \left(\frac{dx}{dt}, \frac{dy}{dt}\right) = (1, -2t) = i - 2xj$. An alternative way is to use the gradient vector of the function $F(x,y) = y + x^2$ which is $\nabla F = (2x, 1)$ and find a vector $\mathbf{v} = (1, -2x)$ which is orthogonal to the gradient vector. The length of the tangent vector, the speed, is $\|\mathbf{v}\| = \sqrt{1+4x^2}$. The directional derivative of f(x,y) is

$$\begin{aligned} D_v f(x,y) &= \nabla f(x,y) \cdot \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= (y,x) \cdot \frac{(1,-2x)}{\sqrt{1+4x^2}} \\ &= \frac{y - 2x^2}{\sqrt{1+4x^2}} = \frac{4 - 3x^2}{\sqrt{1+4x^2}} \end{aligned}$$

From this we can immediately deduce that $D_v f(x,y) > 0$ exactly when $4 - 3x^2 > 0$ or $|x| < \sqrt{\frac{4}{3}}$.

3a).[4 marks] Consider the hyper surface $\mathbf{S} = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 = 6\}$. Find the equation of the tangent plane to the surface $\mathbf{S}$ at the point $(\sqrt{2}, -1, -\sqrt{2}, +1)$

The tangent plane can be found once we have a normal direction. This is provided by the gradient vector for the function whose level set describes the surface $\mathbf{S}$. If $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$ then $\mathbf{S}$ is the level set f=6. The normal direction to this level set at the point $(\sqrt{2}, -1, -\sqrt{2}, +1)$ is $\nabla f((\sqrt{2}, -1, -\sqrt{2}, +1) = (2\sqrt{2}, -2, -2\sqrt{2}, +2)$.
The equation of the tangent plane at the point $(\sqrt{2}, -1, -\sqrt{2}, +1)$ is
$\nabla f \cdot (x - \sqrt{2}, y + 1, z + \sqrt{2}, w - 1) = 0$, or $2\sqrt{2}x - 2y - 2\sqrt{2}z + 2w = 12$

b)[3 marks] Show that the parameterized circle $\vec{\mathbf{r}}(t) = (2\cos(t), -1, -2\sin(t), +1)$ is tangent to the surface $\mathbf{S}$ at the parameter value $t = \frac{\pi}{4}$.

The curve with parameterization $\vec{\mathbf{r}}(t) = (2\cos(t), -1, -2\sin(t), +1)$ intersects the tangent plane at $t = \frac{\pi}{4}$, since $\vec{\mathbf{r}}\left(\frac{\pi}{4}\right) = (\frac{2}{\sqrt{2}}, -1, \frac{-2}{\sqrt{2}}, +1)$ whose components satisfy the equation for the tangent plane.
The condition for tangency of this curve amounts to showing that $\vec{\mathbf{r}}'\left(\frac{\pi}{4}\right) \cdot \nabla f((\sqrt{2}, -1, -\sqrt{2}, +1) = 0$. We calculate this as follows
$(-\frac{2}{\sqrt{2}}, 0, \frac{-2}{\sqrt{2}}, 0) \cdot (2\sqrt{2}, -2, -2\sqrt{2}, +2) = -4 + 4 = 0.$

c)[3 marks] Calculate the derivative of the composite function $f(\vec{\mathbf{r}}(t))$ at $t = \frac{\pi}{4}$ where $f(x, y, z, w) = x^2 + y^2 + z^2 + w^2$.

The composite function can be calculated at general t,
$f(\vec{\mathbf{r}}(t)) = 4\cos^2(t) + 1 + 4\sin^2(t) + 1 = 6$. The composite is constant for all t, therefore $D\left(f(\vec{\mathbf{r}}(t))\right) = \nabla f(\vec{\mathbf{r}}(t)) \cdot \vec{\mathbf{r}}'(t) = 0$. This last equation can also be checked directly at the point $t = \frac{\pi}{4}$.

4a). [4 marks] Let $\vec{F} = F_1\vec{\mathbf{i}} + F_2\vec{\mathbf{j}} = xy\vec{\mathbf{i}} + x^2e^y\vec{\mathbf{j}}$, and let $\mathbf{g}(s,t) = ((s+t), s^2 - t^2)$ denote a change of coordinates in an open domain of $\mathbf{R}^2$. Using the chain rule calculate $\frac{\partial F_2}{\partial s}$ and $\frac{\partial F_1}{\partial t}$ at the point $(s_0, t_0) = (2, 1)$.

We calculate that g(2,1) = (3,3). Using the chain rule evaluated at this point, we find that

$$\begin{aligned}
\frac{\partial F_2}{\partial s} &= \frac{\partial F_2}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial F_2}{\partial y}\frac{\partial y}{\partial s} \\
&= 2xe^y(1) + x^2e^y(4) = 42e^3 \\
\frac{\partial F_1}{\partial t} &= \frac{\partial F_1}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F_1}{\partial y}\frac{\partial y}{\partial t} \\
&= y(1) + x(-2) = -3
\end{aligned}$$

4b). [6 marks] Calculate the derivative of the composite function $D(\vec{F} \circ g)(s_0, t_0)$ at the point $(s_0, t_0) = (2, 1)$.

Using the chain rule again, we find that

$$\begin{aligned}
D(\vec{F} \circ g)(2,1) &= D\left(\vec{F}(3,3)\right) \cdot Dg(2,1) \\
&= \begin{bmatrix} y & x \\ 2xe^y & x^2e^y \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 3 \\ 6e^3 & 9e^3 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 15 & -3 \\ 42e^3 & -12e^3 \end{bmatrix}
\end{aligned}$$

5.a)[4 marks] Using spherical coordinates in three dimensional space, find the polar (azimuth) angle ϕ, and the resulting equation in spherical coordinates of the cone surface $z = 3\sqrt{x^2+y^2}$.

We can use trigonometry to find the polar (azimuth) angle for the cone z=3r. This angle is constant for all points on the cone. For example when r = 1, z=3 and we can use a right angle triangle with hypotenuse $\sqrt{10}$ and polar angle ϕ to determine the value of this angle. In particular slope of the hypotenuse is 3, so

$$\tan(\phi) = \frac{1}{3}, \quad \phi = \arctan\left(\frac{1}{2}\right) = \arccos\left(\frac{3}{\sqrt{10}}\right)$$

b) [6 marks] Consider surface **S** given by the implicit equation $x^3 + y^3 + z^3 - xyz = 0$. Show that (1,0,-1) satisfies this equation, and that we can locally uniquely determine functions y=f(x,z), and z=g(x,y) in a neighborhood of this point, by solving the implicit equation for y or z. Calculate $\frac{\partial y}{\partial z}$ and $\frac{\partial z}{\partial x}$ at the point (1,0,-1).

We cannot solve this equation directly for y or z, so must resort to information supplied by the Implicit function theorem. First, the calculation to show that (1,0,-1) satisfies this equation. This amounts to substitution into the cubic equation and we find 1 -1 = 0 as required.
Next we need to verify that we can locally solve this cubic as functions y=f(x,z), and z=g(x,y) in a neighborhood of this point.
This is accomplished using the implicit function theorem together withthe calculations

$$\begin{aligned}
\frac{\partial F}{\partial y}(1,0,-1) &= 3y^2 - xz \\
&= +1 \neq 0 \\
\frac{\partial F}{\partial z}(1,0,-1) &= 3z^2 - xy \\
&= 3 \neq 0
\end{aligned}$$

Since these partial derivatives are nonzero the implicit function theorem guarantees the local existence of functions y=f(x,z), and z=g(x,y) in a neighborhood of the point (1,0,-1).
Given this crucial information we can now use implicit differentiation to determine the required partial derivatives.

$$\frac{\partial y}{\partial z} = -\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial y}} = -\frac{3}{1}, \qquad \frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{3}{3}$$

Extra page for recording work and answers. Please indicate carefully which questions you are adding material here for.