Solutions #4

1.(a). Let $\vec{F}, \vec{G} \colon \mathbb{R}^n \to \mathbb{R}^3$ be differentiable at $\vec{a} \in \mathbb{R}^n$ and let $\vec{x} \in \mathbb{R}^n$. Show that

$$\big[D(\vec{F} \times \vec{G})(\vec{a})\big]\vec{x} = \big[D\vec{F}(\vec{a})\big]\vec{x} \times \vec{G}(\vec{a}) + \vec{F}(\vec{a}) \times \big[D\vec{G}(\vec{a})\big]\vec{x}\,.$$

Solution. Since both sides of the identity depend linearly on $\vec{x}$, it is sufficient to prove it for $\vec{x} = \vec{e}_1, \vec{x} = \vec{e}_2, \ldots, \vec{x} = \vec{e}_n$. For the rest of the proof we assume that $\vec{x} = \vec{e}_i$.
We first note that multiplying a matrix by the standard basis vector $\vec{e}_i$ simply gives the i-th column of the matrix. Thus, linearity and the product rule yield

$$\begin{aligned}
&\big[D\vec{F}(\vec{a})\big]\vec{e}_i \times \vec{G}(\vec{a}) + \vec{F}(\vec{a}) \times \big[D\vec{G}(\vec{a})\big]\vec{e}_i \\
&= \left(\frac{\partial \vec{F}_1}{\partial x_i}\vec{\imath} + \frac{\partial \vec{F}_2}{\partial x_i}\vec{\jmath} + \frac{\partial \vec{F}_3}{\partial x_i}\vec{k}\right) \times \left(\vec{G}_1\vec{\imath} + \vec{G}_2\vec{\jmath} + \vec{G}_3\vec{k}\right) \\
&\quad + \left(\vec{F}_1\vec{\imath} + \vec{F}_2\vec{\jmath} + \vec{F}_3\vec{k}\right) \times \left(\frac{\partial \vec{G}_1}{\partial x_i}\vec{\imath} + \frac{\partial \vec{G}_2}{\partial x_i}\vec{\jmath} + \frac{\partial \vec{G}_3}{\partial x_i}\vec{k}\right) \\
&= \left(\frac{\partial \vec{F}_2}{\partial x_i}\vec{G}_3 - \frac{\partial \vec{F}_3}{\partial x_i}\vec{G}_2\right)\vec{\imath} + \left(\frac{\partial \vec{F}_3}{\partial x_i}\vec{G}_1 - \frac{\partial \vec{F}_1}{\partial x_i}\vec{G}_3\right)\vec{\jmath} + \left(\frac{\partial \vec{F}_1}{\partial x_i}\vec{G}_2 - \frac{\partial \vec{F}_2}{\partial x_i}\vec{G}_1\right)\vec{k} \\
&\quad + \left(\vec{F}_2\frac{\partial \vec{G}_3}{\partial x_i} - \vec{F}_3\frac{\partial \vec{G}_2}{\partial x_i}\right)\vec{\imath} + \left(\vec{F}_3\frac{\partial \vec{G}_1}{\partial x_i} - \vec{F}_1\frac{\partial \vec{G}_3}{\partial x_i}\right)\vec{\jmath} + \left(\vec{F}_1\frac{\partial \vec{G}_2}{\partial x_i} - \vec{F}_2\frac{\partial \vec{G}_1}{\partial x_i}\right)\vec{k} \\
&= \frac{\partial}{\partial x_i}\left(\vec{F}_2\vec{G}_3 - \vec{F}_3\vec{G}_2\right)\vec{\imath} + \frac{\partial}{\partial x_i}\left(\vec{F}_3\vec{G}_1 - \vec{F}_1\vec{G}_3\right)\vec{\jmath} + \frac{\partial}{\partial x_i}\left(\vec{F}_1\vec{G}_2 - \vec{F}_2\vec{G}_1\right)\vec{k} \\
&= \frac{\partial}{\partial x_i}\Big((\vec{F} \times \vec{G})(\vec{a})\Big) \\
&= \big[D(\vec{F} \times \vec{G})(\vec{a})\big]\vec{e}_i\,.
\end{aligned}$$

□

1.(b). Suppose that $n = 3$, $\vec{F}(\vec{a}) = 2\vec{\imath} + \vec{\jmath} + 2\vec{k}$, $\vec{G}(\vec{a}) = \vec{\imath} + 2\vec{\jmath} + \vec{k}$,

$$D\vec{F}(\vec{a}) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D\vec{G}(\vec{a}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Find $\big[D(\vec{F} \times \vec{G})(\vec{a})\big](\vec{\imath} + \vec{\jmath} + \vec{k})$.

Solution. Linearity and part (a) give

$$\begin{aligned}
\big[D(\vec{F} \times \vec{G})(\vec{a})\big](\vec{\imath} + \vec{\jmath} + \vec{k}) &= \big[D\vec{F}(\vec{a})\big](\vec{\imath} + \vec{\jmath} + \vec{k}) \times \vec{G}(\vec{a}) + \vec{F}(\vec{a}) \times \big[D\vec{G}(\vec{a})\big](\vec{\imath} + \vec{\jmath} + \vec{k}) \\
&= (\vec{\imath} + 2\vec{\jmath} + 3\vec{k}) \times (\vec{\imath} + \vec{\jmath} + \vec{k}) + (\vec{\imath} + 2\vec{\jmath} + \vec{k}) \times (-\vec{\imath}) \\
&= (-4\vec{\imath} + 2\vec{\jmath}) + (-2\vec{\imath} + \vec{k}) \\
&= -4\vec{\imath} + \vec{k}\,.
\end{aligned}$$

□

2.(a). Two surfaces are said to be ***orthogonal*** to each other at a point P if the normals to their tangent planes are perpendicular at P. Show that the surfaces

$$z = \tfrac{1}{2}(x^2 + y^2 - 1) \quad \text{and} \quad z = \tfrac{1}{2}(1 - x^2 - y^2)$$

are orthogonal at all points of intersection.

Solution. If $f(x, y, z) = \frac{1}{2}(x^2 + y^2 - 1) - z$ and $g(x, y, z) = \frac{1}{2}(1 - x^2 - y^2) - z$, then the normals to the tangent planes of the level surfaces $f(x, y, z) = 0$ and $g(x, y, z) = 0$ are $\vec{\boldsymbol{\nabla}} f(x, y, z) = x\vec{\imath} + y\vec{\jmath} - \vec{\boldsymbol{k}}$ and $\vec{\boldsymbol{\nabla}} g(x, y, z) = -x\vec{\imath} - y\vec{\jmath} - \vec{\boldsymbol{k}}$ respectively. On the other hand, the surfaces $z = \frac{1}{2}(x^2 + y^2 - 1)$ and $z = \frac{1}{2}(1 - x^2 - y^2)$ intersect at the point (x, y, z) if and only if $\frac{1}{2}(x^2 + y^2 - 1) = \frac{1}{2}(1 - x^2 - y^2)$ or $1 - x^2 - y^2 = 0$. Hence, $\vec{\boldsymbol{\nabla}} f \cdot \vec{\boldsymbol{\nabla}} g = -x^2 - y^2 + 1 = 0$ at all points of intersection and the surfaces are orthogonal. □

2.(b). Show that the Laplacian operator $\nabla^2 := \vec{\boldsymbol{\nabla}} \cdot \vec{\boldsymbol{\nabla}} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$ in $\mathbb{R}^2$ is given in polar coordinates by the formula

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

Solution. Let f denote a smooth function of x and y. Since $x = r\cos(\theta)$ and $y = r\sin(\theta)$, the chain rule yields

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \cos(\theta)\frac{\partial f}{\partial x} + \sin(\theta)\frac{\partial f}{\partial y} &&\Rightarrow \quad \frac{\partial}{\partial r} = \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y} &&\Rightarrow \quad \frac{\partial}{\partial \theta} = -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y}\,.
\end{aligned}$$

Further applications of the chain rule also give

$$\begin{aligned}
\frac{\partial^2 f}{\partial r^2} &= \frac{\partial}{\partial r}\left(\frac{\partial f}{\partial r}\right) = \frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial f}{\partial x} + \sin(\theta)\frac{\partial f}{\partial y}\right) = \cos(\theta)\frac{\partial^2 f}{\partial r \partial x} + \sin(\theta)\frac{\partial^2 f}{\partial r \partial y} \\
&= \cos(\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y \partial x}\frac{\partial y}{\partial r}\right) + \sin(\theta)\left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial r}\right) \\
&= \cos^2(\theta)\frac{\partial^2 f}{\partial x^2} + 2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x \partial y} + \sin^2(\theta)\frac{\partial^2 f}{\partial y^2} \\
\Rightarrow \qquad \frac{\partial^2}{\partial r^2} &= \cos^2(\theta)\frac{\partial^2}{\partial x^2} + 2\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial x \partial y} + \sin^2(\theta)\frac{\partial^2}{\partial y^2}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial\theta^2} &= \frac{\partial}{\partial\theta}\left(\frac{\partial f}{\partial\theta}\right) = \frac{\partial}{\partial\theta}\left(-r\sin(\theta)\frac{\partial f}{\partial x} + r\cos(\theta)\frac{\partial f}{\partial y}\right) \\
&= -r\cos(\theta)\frac{\partial f}{\partial x} - r\sin(\theta)\frac{\partial^2 f}{\partial\theta\partial x} - r\sin(\theta)\frac{\partial f}{\partial y} + r\cos(\theta)\frac{\partial^2 f}{\partial\theta\partial y} \\
&= -r\left(\cos(\theta)\frac{\partial f}{\partial x} + \sin(\theta)\frac{\partial f}{\partial y}\right) - r\sin(\theta)\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y\partial x}\frac{\partial y}{\partial\theta}\right) \\
&\quad + r\cos(\theta)\left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial x}{\partial\theta} + \frac{\partial^2 f}{\partial y^2}\frac{\partial y}{\partial\theta}\right) \\
&= -r\frac{\partial f}{\partial r} + r^2\sin^2(\theta)\frac{\partial^2 f}{\partial x^2} - 2r^2\sin(\theta)\cos(\theta)\frac{\partial^2 f}{\partial x\partial y} + r^2\sin^2(\theta)\frac{\partial^2 f}{\partial y^2}
\end{aligned}
$$

$$
\Rightarrow \qquad \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r}\frac{\partial}{\partial r} = \sin^2(\theta)\frac{\partial^2}{\partial x^2} - 2\sin(\theta)\cos(\theta)\frac{\partial^2}{\partial x\partial y} + \cos^2(\theta)\frac{\partial^2}{\partial y^2}\,.
$$

Since $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r} = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$. $\square$

Alternative Solution for 2(b). Rearranging the first two equations from the solution for 2(b), we obtain $\dfrac{\partial}{\partial x} = \cos(\theta)\dfrac{\partial}{\partial r} - \dfrac{\sin(\theta)}{r}\dfrac{\partial}{\partial\theta}$ and $\dfrac{\partial}{\partial y} = \sin(\theta)\dfrac{\partial}{\partial r} + \dfrac{\cos(\theta)}{r}\dfrac{\partial}{\partial\theta}$. Combining these with linearity and the product rule yield

$$
\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right)\left(\cos(\theta)\frac{\partial}{\partial r} - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right) \\
&= \cos(\theta)\frac{\partial}{\partial r}\left(\cos(\theta)\frac{\partial}{\partial r}\right) - \cos(\theta)\frac{\partial}{\partial r}\left(\frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right) - \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\left(\cos(\theta)\frac{\partial}{\partial r}\right) \\
&\quad + \frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\left(\frac{\sin(\theta)}{r}\frac{\partial}{\partial\theta}\right) \\
&= \cos(\theta)(0)\frac{\partial}{\partial r} + \cos^2(\theta)\frac{\partial^2}{\partial r^2} + \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial\theta} - \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial\theta} + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} \\
&\quad - \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial\theta} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&= \cos^2(\theta)\frac{\partial^2}{\partial r^2} - \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial\theta} - \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial\theta\partial r} + \frac{\sin^2(\theta)}{r^2}\frac{\partial^2}{\partial\theta^2} \\
&\quad + \frac{\sin^2(\theta)}{r}\frac{\partial}{\partial r} + \frac{2\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial\theta}
\end{aligned}
$$

and

$$\begin{aligned}
\frac{\partial^2}{\partial y^2} &= \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = \left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right)\left(\sin(\theta)\frac{\partial}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right) \\
&= \sin(\theta)\frac{\partial}{\partial r}\left(\sin(\theta)\frac{\partial}{\partial r}\right) + \sin(\theta)\frac{\partial}{\partial r}\left(\frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right) + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial}{\partial r}\right) \\
&\quad + \frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\left(\frac{\cos(\theta)}{r}\frac{\partial}{\partial \theta}\right) \\
&= \sin(\theta)(0)\frac{\partial}{\partial r} + \sin^2(\theta)\frac{\partial^2}{\partial r^2} - \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} \\
&\quad + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial \theta\partial r} - \frac{\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2} \\
&= \sin^2(\theta)\frac{\partial^2}{\partial r^2} + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial r\partial \theta} + \frac{\cos(\theta)\sin(\theta)}{r}\frac{\partial^2}{\partial \theta\partial r} + \frac{\cos^2(\theta)}{r^2}\frac{\partial^2}{\partial \theta^2} \\
&\quad + \frac{\cos^2(\theta)}{r}\frac{\partial}{\partial r} - \frac{2\cos(\theta)\sin(\theta)}{r^2}\frac{\partial}{\partial \theta}\,.
\end{aligned}$$

Since $\cos^2(\theta) + \sin^2(\theta) = 1$, we obtain $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = \dfrac{\partial^2}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial \theta^2} + \dfrac{1}{r}\dfrac{\partial}{\partial r}$. □

3. Consider the function

$$f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Find the partial derivatives $f_x(0,0)$ and $f_y(0,0)$.

(b) If $\vec{\boldsymbol{H}}\colon \mathbb{R} \to \mathbb{R}^2$ is defined by $\vec{\boldsymbol{H}}(t) = at\vec{\imath} + bt\vec{\jmath}$ for constants a and b, then show that $f \circ \vec{\boldsymbol{H}}$ is differentiable and find $D(f \circ \vec{\boldsymbol{H}})(0)$.

(c) Calculate $Df(0,0)D\vec{\boldsymbol{H}}(0)$. How can this answer be reconciled with the answer in part (b) and the chain rule?

Solution.

(a) The definition of a partial derivative implies that

$$\begin{aligned}
f_x(0,0) &= \lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0 \\
f_y(0,0) &= \lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0\,.
\end{aligned}$$

(b) If $g\colon \mathbb{R} \to \mathbb{R}$ is given by $g(t) = (f \circ \vec{\boldsymbol{H}})(t)$ then

$$g(t) = \begin{cases} \left(\dfrac{ab^2}{a^2+b^2}\right)t & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

For $t \neq 0$, g is a linear function and hence differentiable. For $t = 0$, the definition of the derivative (for a function of a single variable) gives

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t} = \lim_{t \to 0} \frac{\left(\frac{ab^2}{a^2+b^2}\right)t - 0}{t} = \frac{ab^2}{a^2 + b^2}\,.$$

Thus, g is differentiable on $\mathbb{R}$ and $g'(0) = D(f \circ \vec{\boldsymbol{H}})(0) = \frac{ab^2}{a^2+b^2}$.

(c) Part (a) implies that $Df(0,0) = \begin{bmatrix} 0 & 0 \end{bmatrix}$. Since $D\vec{\boldsymbol{H}}(t) = \begin{bmatrix} a \\ b \end{bmatrix}$, we have

$$Df(0,0)Dg(0) = \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = 0\,.$$

Since $\vec{\boldsymbol{H}}$ is a smooth function on $\mathbb{R}$ and $D(f \circ \vec{\boldsymbol{H}})(0) \neq Df(0,0)D\vec{\boldsymbol{H}}(0)$, the Chain rule implies that f is not differentiable at the origin. $\square$

Remark. We could also show directly that f is not differentiable at the origin. Since $f_x(0,0) = 0 = f_y(0,0)$, the candidate for a "best" linear approximation to f near the origin is $\ell(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0$. The function f is differentiable provide the linear approximation ℓ satisfies the small relative error criterion: $\lim\limits_{(x,y)\to(0,0)} \frac{f(x,y)-\ell(x,y)}{\|(x,y)\|} = 0$. If this limits exists, then we get the same value no matter how (x,y) approaches $(0,0)$. However, when $x = y$, we have

$$\lim_{(x,y)\to(0,0)} \frac{f(x,y) - \ell(x,y)}{\|(x,y)\|} = \lim_{x \to 0} \frac{f(x,x) - 0}{\|(x,x)\|} = \lim_{x \to 0} \frac{\frac{x(x)^2}{x^2+x^2}}{\sqrt{x^2 + x^2}} = \frac{1}{2\sqrt{2}} \neq 0\,.$$

This shows that f is not differentiable at the origin.