Solutions #5

1. Consider the surface defined by the equation

$$x^3 z + x^2 y^2 + \sin(yz) = -3\,.$$

(a) Find an equation for the plane tangent to this surface at the point $(-1, 0, 3)$.
(b) Parametrize the line normal to this surface at the point $(-1, 0, 3)$.

Solution.

(a) If $f(x, y, z) = x^3 z + x^2 y^2 + \sin(yz)$, then we have

$$\vec{\boldsymbol{\nabla}} f(x, y, z) = (3x^2 z + 2xy^2)\vec{\boldsymbol{\imath}} + \big(2x^2 y + z\cos(yz)\big)\vec{\boldsymbol{\jmath}} + \big(x^3 + y\cos(yz)\big)\vec{\boldsymbol{k}}\,.$$

Hence, the equation for the plane tangent to the surface at the point $(-1, 0, 3)$ is

$$\begin{aligned} 0 &= \vec{\boldsymbol{\nabla}} f(-1, 0, 3) \cdot \big((x\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + z\vec{\boldsymbol{k}}) - (-1\vec{\boldsymbol{\imath}} + 3\vec{\boldsymbol{k}})\big) \\ &= (9\vec{\boldsymbol{\imath}} + 3\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}) \cdot \big((x+1)\vec{\boldsymbol{\imath}} + y\vec{\boldsymbol{\jmath}} + (z-3)\vec{\boldsymbol{k}}\big) \\ &= 9x + 9 + 3y - z + 3 \end{aligned}$$

or simply $9x + 3y - z = -12$.

(b) From part (a), we deduce that the vector $\vec{\boldsymbol{n}} := 9\vec{\boldsymbol{\imath}} + 3\vec{\boldsymbol{\jmath}} - \vec{\boldsymbol{k}}$ is normal to the surface at $\vec{\boldsymbol{p}} := (-1, 0, 3)$. Thus, the function $\vec{\boldsymbol{\ell}}(t) := \vec{\boldsymbol{n}}t + \vec{\boldsymbol{p}} = (9t - 1)\vec{\boldsymbol{\imath}} + 3t\vec{\boldsymbol{\jmath}} + (3 - t)\vec{\boldsymbol{k}}$ parametrizes the line normal to this surface that the given point. □

2. Consider the path $\vec{\boldsymbol{\beta}}\colon (0, \pi) \to \mathbb{R}^2$ given by $\vec{\boldsymbol{\beta}}(t) := \sin(t)\vec{\boldsymbol{\imath}} + \big(\cos(t) + \ln(\tan(t/2))\big)\vec{\boldsymbol{\jmath}}$. The underlying curve is called the ***tractrix***.

(a) Show that derivative $\vec{\boldsymbol{\beta}}'(t)$ is nonzero at everywhere except $t = \pi/2$.
(b) Show that the length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1.

Solution.

(a) Since $0 < t < \pi$ and

$$\begin{aligned} \vec{\boldsymbol{\beta}}'(t) &= \cos(t)\vec{\boldsymbol{\imath}} + \left(-\sin(t) + \frac{1}{\tan(t/2)}\frac{1}{\cos^2(t/2)}\frac{1}{2}\right)\vec{\boldsymbol{\jmath}} \\ &= \cos(t)\vec{\boldsymbol{\imath}} + \left(-\sin(t) + \frac{1}{2\sin(t/2)\cos(t/2)}\right)\vec{\boldsymbol{\jmath}} = \cos(t)\vec{\boldsymbol{\imath}} + \left(-\sin(t) + \frac{1}{\sin(t)}\right)\vec{\boldsymbol{\jmath}} \\ &= \cos(t)\vec{\boldsymbol{\imath}} + \left(\frac{1 - \sin^2(t)}{\sin(t)}\right)\vec{\boldsymbol{\jmath}} = \cos(t)\left[\vec{\boldsymbol{\imath}} + \left(\frac{\cos(t)}{\sin(t)}\right)\vec{\boldsymbol{\jmath}}\right], \end{aligned}$$

we see that $\vec{\boldsymbol{\beta}}'(t) = \vec{\boldsymbol{0}}$ if and only if $\cos(t) = 0$ or equivalently $t = \pi/2$.

(b) For $t \neq \pi/2$, the tangent line of the tractrix through the point $\vec{\boldsymbol{\beta}}(t)$ is parametrized by the path $\vec{\boldsymbol{\ell}}\colon \mathbb{R} \to \mathbb{R}^2$ where

$$\vec{\boldsymbol{\ell}}(u) := \vec{\boldsymbol{\beta}}'(t)u + \vec{\boldsymbol{\beta}}(t) = \big(\cos(t)u + \sin(t)\big)\vec{\boldsymbol{\imath}} + \left(\frac{\cos^2(t)}{\sin(t)}u + \cos(t) + \ln(\tan(t/2))\right)\vec{\boldsymbol{\jmath}}.$$

By construction, this line meets the tractrix at the point $\vec{\boldsymbol{\beta}}(t)$ or $u = 0$. This line meets the y-axis when $\cos(t)u + \sin(t) = 0$ or $u = -\tan(t)$. When $0 < t < \pi/2$, we have $-\tan(t) < 0$; when $\pi/2 < t < \pi$, we have $-\tan(t) > 0$. Hence, length of the segment of the tangent line between the point of tangency and the y-axis is

$$\begin{aligned} \text{length} &= \left| \int_{-\tan(t)}^{0} \|\vec{\boldsymbol{\ell}}\,'(u)\| \, du \right| = \left| \int_{-\tan(t)}^{0} \|\vec{\boldsymbol{\beta}}\,'(t)\| \, du \right| = \|\vec{\boldsymbol{\beta}}\,'(t)\| \left| \int_{-\tan(t)}^{0} du \right| \\ &= \|\vec{\boldsymbol{\beta}}\,'(t)\| |\tan(t)| = \left(|\cos(t)| \sqrt{1 + \frac{\cos^2(t)}{\sin^2(t)}} \right) |\tan(t)| \\ &= \left(|\cos(t)| \sqrt{\frac{\sin^2(t) + \cos^2(t)}{\sin^2(t)}} \right) |\tan(t)| = \left| \frac{\cos(t)}{\sin(t)} \tan(t) \right| = 1 \,. \end{aligned}$$

Therefore, the length of the segment of the tangent of the tractrix between the point of tangency and the y-axis is constantly equal to 1. □

Alternative Solution to 2(b). Since the distance between two points is given by a line, one can also compute the length of the segment of the tangent line between the point of tangency and the y-axis as follows:

$$\begin{aligned} \text{length} &= \left\| \vec{\boldsymbol{\ell}}\big(-\tan(t)\big) - \vec{\boldsymbol{\ell}}(0) \right\| = \left\| \vec{\boldsymbol{\beta}}(t) - \tan(t) \vec{\boldsymbol{\beta}}\,'(t) - \vec{\boldsymbol{\beta}}(t) \right\| = |\tan(t)| \|\vec{\boldsymbol{\beta}}\,'(t)\| \\ &= |\tan(t)| |\cos(t)| \left\| \vec{\boldsymbol{\imath}} + \frac{\cos(t)}{\sin(t)} \vec{\boldsymbol{\jmath}} \right\| = |\sin(t)| \sqrt{1 + \frac{\cos^2(t)}{\sin^2(t)}} = \sqrt{\sin^2(t) + \cos^2(t)} = 1 \,. \square \end{aligned}$$

3(a). Show that the differentiable path $\vec{\boldsymbol{\gamma}} \colon \mathbb{R} \setminus \{\vec{\mathbf{0}}\} \to \mathbb{R}^3$ given by

$$\vec{\boldsymbol{\gamma}}(t) = e^{2t} \vec{\boldsymbol{\imath}} + \ln|t| \vec{\boldsymbol{\jmath}} + \tfrac{1}{t} \vec{\boldsymbol{k}}$$

is a flow line of the vector field $\vec{\boldsymbol{F}} \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\vec{\boldsymbol{F}}(x, y, z) = 2x \vec{\boldsymbol{\imath}} + z \vec{\boldsymbol{\jmath}} - z^2 \vec{\boldsymbol{k}}$.

Solution. The path $\vec{\boldsymbol{\gamma}}$ is a flow line of $\vec{\boldsymbol{F}}$ if and only if it satisfies the following system of differential equations:

$$\vec{\boldsymbol{\gamma}}\,'(t) = \vec{\boldsymbol{F}}\big(\vec{\boldsymbol{\gamma}}(t)\big) \qquad \Longleftrightarrow \qquad \begin{cases} \vec{\gamma}_1'(t) = 2\vec{\gamma}_1(t) \\ \vec{\gamma}_2'(t) = \vec{\gamma}_3(t) \\ \vec{\gamma}_3'(t) = -\big(\vec{\gamma}_3(t)\big)^2 \end{cases} .$$

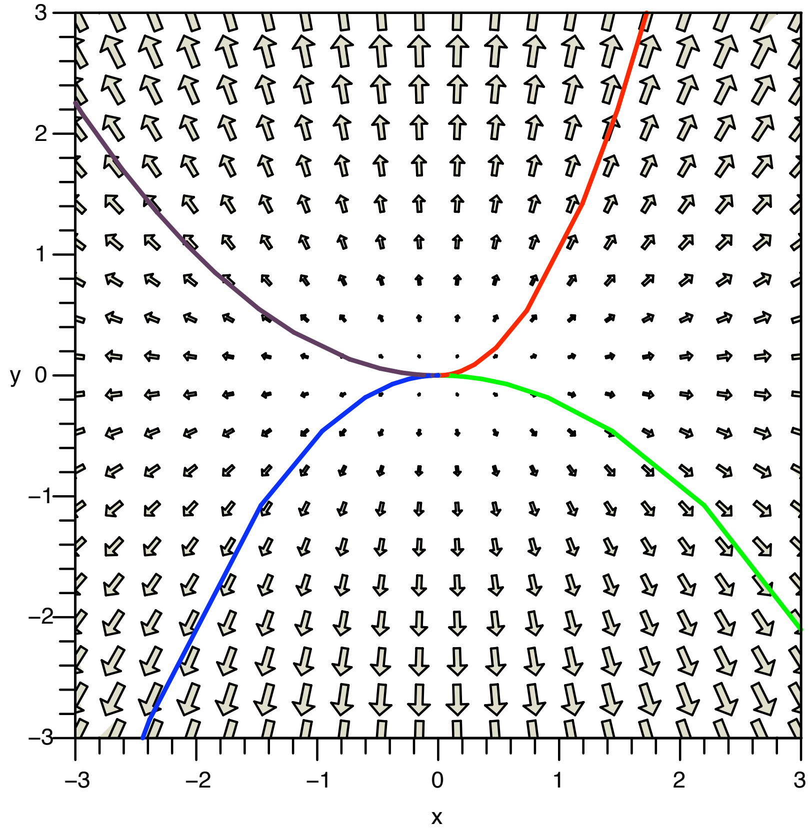
Since we have

$$\begin{aligned} \vec{\gamma}_1'(t) &= \frac{d}{dt}\big(e^{2t}\big) = 2e^{2t} = 2\vec{\gamma}_1(t) \\ \vec{\gamma}_2'(t) &= \frac{d}{dt}\big(\ln|t|\big) = \frac{1}{t} = \vec{\gamma}_3(t) \\ \vec{\gamma}_3'(t) &= \frac{d}{dt}\left(\frac{1}{t}\right) = -\frac{1}{t^2} = -\big(\vec{\gamma}_3(t)\big)^2 , \end{aligned}$$

we conclude that $\vec{\boldsymbol{\gamma}}$ is a flow line of $\vec{\boldsymbol{F}}$. $\square$

3(b). Find the flow lines of the vector field $\vec{\boldsymbol{G}} \colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $\vec{\boldsymbol{G}}(x, y) = x\vec{\boldsymbol{\imath}} + 2y\vec{\boldsymbol{\jmath}}$.

Solution. The following figure represents the vector field and four distinct flow lines.



Let $\vec{\boldsymbol{\alpha}} \colon \mathbb{R} \to \mathbb{R}^2$ be a differentiable path. The path $\vec{\boldsymbol{\alpha}}(t) = \vec{\boldsymbol{\alpha}}_1(t)\vec{\boldsymbol{\imath}} + \vec{\boldsymbol{\alpha}}_2(t)\vec{\boldsymbol{\jmath}}$ is a flow line of $\vec{\boldsymbol{G}}$ if and only if it satisfies the following system of differential equations:

$$\vec{\boldsymbol{\alpha}}\,'(t) = \vec{\boldsymbol{G}}\big(\vec{\boldsymbol{\alpha}}(t)\big) \qquad \Longleftrightarrow \qquad \begin{cases} \vec{\boldsymbol{\alpha}}_1'(t) = \vec{\boldsymbol{\alpha}}_1(t) \\ \vec{\boldsymbol{\alpha}}_2'(t) = 2\vec{\boldsymbol{\alpha}}_2(t) \end{cases}$$

Solving this pair of ordinary differential equations, we see that $\vec{\boldsymbol{\alpha}}_1(t) = c_1 e^t$ and $\vec{\boldsymbol{\alpha}}_2(t) = c_2 e^{2t}$ where $\vec{\boldsymbol{c}} := (c_1, c_2) \in \mathbb{R}^2$ is constant. The flow line $\vec{\boldsymbol{\alpha}}(t) = c_1 e^t \vec{\boldsymbol{\imath}} + c_2 e^{2t} \vec{\boldsymbol{\jmath}}$ passes through the point (c_1, c_2) at $t = 0$. Observe that all flow line converge at the origin as t tends to $-\infty$. $\square$