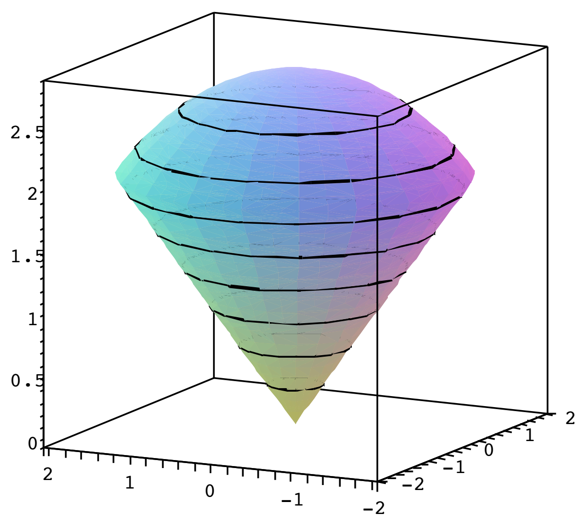
Solutions #8

1. **(a)** Find the volume of an ice cream cone bounded by the cone $z = \sqrt{x^2 + y^2}$ and the hemisphere $z = \sqrt{8 - x^2 - y^2}$.

(b) Find the average distance to the origin for points in the ice cream cone region bounded by the hemisphere $z = \sqrt{8 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$.

Solution.

(a) The cone meets the hemisphere when $\sqrt{x^2 + y^2} = \sqrt{8 - x^2 - y^2}$. In polar coordinates, this equation becomes $r = \sqrt{8 - r^2} \iff 2r^2 = 8 \iff r = 2$. Hence, we can compute the volume of the ice cream cone by finding the volume under the graph of $\sqrt{8 - r^2}$ above the disk $R = \big\{(r, \theta) : 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\big\}$ and subtracting the volume



under the graph of r above R. Therefore, we have

$$
\begin{aligned}
\text{Volume} &= \int_0^{2\pi} \int_0^2 \big(\sqrt{8 - r^2}\big)\, r dr\, d\theta - \int_0^{2\pi} \int_0^2 \big(r\big)\, r dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 r\sqrt{8 - r^2} - r^2 \, dr\, d\theta = \int_0^{2\pi} \left[-\frac{1}{3}(8 - r^2)^{3/2} - \frac{1}{3}r^3\right]_0^2 d\theta \\
&= \frac{1}{3}\int_0^{2\pi} \left(-4^{3/2} - 8 + 8^{3/2}\right)\, d\theta = \frac{1}{3}(16\sqrt{2} - 16)\int_0^{2\pi} d\theta \\
&= \frac{1}{3}(32\pi)(\sqrt{2} - 1)\,.
\end{aligned}
$$

(b) In spherical coordinates, the hemisphere is given by

$$\begin{aligned}\rho\cos(\phi) &= \sqrt{8-\rho^2\sin^2(\phi)\cos^2(\theta)-\rho^2\sin^2(\phi)\sin^2(\theta)}\\ \rho^2\cos^2(\phi) &= 8-\rho^2\sin^2(\phi)\\ \rho &= 2\sqrt{2}\end{aligned}$$

and the cone is given by the equation

$$\begin{aligned}\rho\cos(\phi) &= \sqrt{\rho^2\sin^2(\phi)\cos^2(\theta)+\rho^2\sin^2(\phi)\sin^2(\theta)}\\ \rho\cos(\phi) &= \rho\sin(\phi)\\ \tan(\phi) &= 1 \quad\Longrightarrow\quad \phi=\frac{\pi}{4}\,.\end{aligned}$$

Hence, the ice cream cone is the region

$$W := \{(\rho,\theta,\phi) : 0\le\rho\le 2\sqrt{2}, 0\le\theta\le 2\pi, 0\le\phi\le\pi/4\}\,.$$

Since the function which measures the distance from the origin to a point is simply ρ, the average distance to the origin for points in W is:

$$\begin{aligned}\text{Average} &= \frac{1}{\text{Volume}(W)}\int_W \rho\,dV = \frac{1}{\text{Volume}(W)}\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho\,\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta\\ &= \frac{1}{\text{Volume}(W)}\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{1}{4}\rho^4\sin(\phi)\right]_0^{2\sqrt{2}} d\phi\,d\theta\\ &= \frac{32\pi}{\text{Volume}(W)}\big[-\cos(\phi)\big]_0^{\pi/4} = \frac{32\pi\left(1-\frac{1}{\sqrt{2}}\right)}{\text{Volume}(W)}\end{aligned}$$

Moreover, we have [also see part (a)]:

$$\text{Volume}(W) = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2\sqrt{2}} \rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta = \frac{32\pi(\sqrt{2}-1)}{3}\,.$$

Therefore, the average distance to the origin for points in the ice cream cone region R is $3/\sqrt{2}$. □

2(a). A bead is made by drilling a cylindrical hole of radius 1 mm through a sphere of radius 5 mm. Set up a triple integral in cylindrical coordinates representing the volume of the bead. Evaluate the integral.

Solution. In cylindrical coordinates, the sphere is given by the equation $r^2+z^2=25$ and the hole is given by $r=1$. Hence, the bead is the region

$$B := \{(r,\theta,z) : -\sqrt{25-r^2}\le z\le\sqrt{25-r^2}, 0\le\theta\le 2\pi, 1\le r\le 5\}\,.$$

We find the volume by integrating the constant density function 1 over B:

$$\text{Volume} = \int_B 1\, dV = \int_1^5 \int_0^{2\pi} \int_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} r\, dz\, d\theta\, dr = \int_1^5 \int_0^{2\pi} \Big[rz\Big]_{-\sqrt{25-r^2}}^{\sqrt{25-r^2}} d\theta\, dr$$

$$= \int_1^5 2r\sqrt{25-r^2} \int_0^{2\pi} d\theta\, dr = 2\pi\left[-\frac{2}{3}(25-r^2)^{3/2}\right]_1^5 = 64\pi\sqrt{6}\ \text{mm}^3\,. \qquad \square$$

2(b). Use the change of variables $x = u - uv$, $y = uv$, to calculate $\int_R \frac{1}{x+y}\, dy\, dx$ where R is the region bounded by $x = 0$, $y = 0$, $x + y = 1$ and $x + y = 4$.

Solution. The change of variables $x = u - uv$, $y = uv$ maps the lines $x = 0$, $y = 0$, $x + y = 1$ and $x + y = 4$ into $u(1-v) = 0$, $uv = 0$, $u = 1$ and $u = 4$ respectively. Hence, the region R corresponds to the region $S = \big\{(u,v) : 1 \le u \le 4, 0 \le v \le 1\big\}$. Since

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{bmatrix} = \begin{bmatrix} 1-v & -u \\ v & u\end{bmatrix} = (1-v)u + uv = u\,,$$

we have

$$\int_R \frac{1}{x+y}\, dA = \int_S \frac{1}{u}|u|\, dv\, du = \int_1^4 \int_0^1 dv\, du = (4-1)(1) = 3\,. \qquad \square$$

3.

(a) By changing to polar coordinates, evaluate the integral

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA.$$

(b) Show that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Solution. **(a)** Changing to polar coordinates we calculate

$$\begin{aligned}\textstyle\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA &= \textstyle\int_0^\infty \int_0^{2\pi} e^{-r^2} r\, d\theta dr = \int_0^\infty e^{-r^2} r\theta|_{\theta=0}^{\theta=2\pi}\, dr \\ &= \textstyle 2\pi \int_0^\infty \frac{1}{2} e^{-r^2} d(r^2) = \pi(-e^{-r^2})|_{r=0}^{r=\infty} = \pi.\end{aligned}$$

(b) Using the following identity (Can you prove it?):

$$\int_a^b \int_c^d f(x)g(y)\, dydx = (\int_a^b f(x)\, dx)(\int_c^d g(y)\, dy),$$

we obtain

$$\int_{\mathbb{R}^2} e^{-(x^2+y^2)}\, dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2} e^{-y^2}\, dydx = (\int_{-\infty}^{\infty} e^{-x^2}\, dx)(\int_{-\infty}^{\infty} e^{-y^2}\, dy) = I^2,$$

where $I = \int_{-\infty}^{\infty} e^{-x^2}\, dx$. Now, combining with (a) we conclude that $I^2 = \pi$, which implies that $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$. $\square$