1.5 The punctured tire

The graph

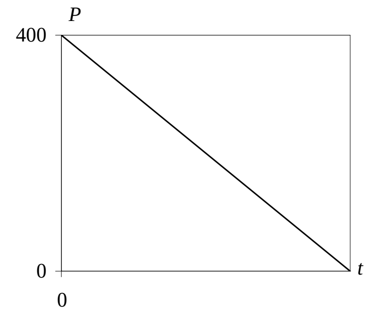
You have a hole in your tire. You pump it up to P=400 kilopascals (kPa) and over the next few hours it goes down till the tire is quite flat. Draw what you think the graph of tire pressure P against time t should look like.

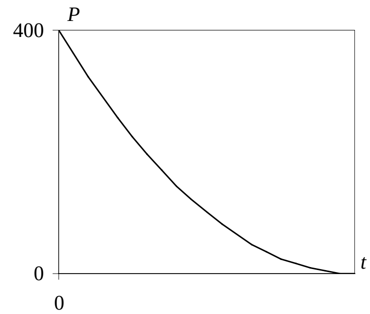
I throw this question out to the class and someone (bless his heart) comes up and draws a straight line. That's good because that engenders a debate and after a bit of argument the class settles on the concave-up curve at the right. As air flows out, P decreases, but as the pressure in the tire gets less, there's "less force" pushing the air out and so the air will flow out more slowly, so therefore the pressure will go down more slowly. Eventually the pressure drops to zero and the graph hits the t-axis.

The problem I give them now is to construct a model for the P-t relationship. That is, they are required to use what understanding of air pressure they already have, or can acquire, to determine exactly how the pressure in the leaking tire should change over time, and hence find the form of an equation relating P to t. Somehow, they have to make that "less force" idea precise.

If I ask the class what sort of equation they think it might be, they suggest that maybe it's a parabola. [This in spite of the fact that this is the exponential section? Go figure.] There are a couple of reasons for that. One is that it's still the curve they are most comfortable with. The other is that some of them have done the water tank experiment, either in this course (see chapter 3) or in Grade 11, and they got a parabola out of that, and isn't this really the same thing?

That's actually a good question and one I remember being stuck on in my green and youthful days. A water tank with a hole seems very similar to a tire with a hole, with water instead of air. Shouldn't they give the same kind of curve?





Relative and absolute pressure

When you put your pressure gauge on the tire you read 400 kPa and you think that's the pressure inside but you're wrong. It's actually 500 kPa. The gauge measures pressure <u>relative</u> to the outside air, and ambient air pressure is 100 kPa. If you took the same tire into outer space, the gauge would read 500 kPa.

Here we always work with relative pressure. That's what everybody always does.

But let me issue a warning. The theoretical arguments we are about to make, really only seem to apply to a tire that sits in a vacuum, with air escaping into an empty space. Well that's the place to start—the argument is simplest in that case—a 400 kPa tire in a 0 kPa space.

It turns out that the very same argument works, with a slight shift of meaning, for a 500 kPa tire in a 100kPa space. More on this later.

Is a tire the same as a water tank?

Let's start with the water tank. The water gets pushed out of the hole because of the pressure, and that's determined by the amount of the water in the tank, and as the water flows out the pressure gets less so the flow rate decreases. Now that's what happens in the tire as well. The air gets pushed out because of the pressure, and that's determined by the amount of air in the tire, and as the air flows out the pressure gets less so the flow rate decreases. There seems to be a big similarity here.

I lead my students through this analysis with as little prompting as possible. In fact I am impressed by how much they seem to be able to do, at least the ones who are prepared to contribute.

But it turns out that water and air are different in an essential way. Water pours out of a hole in the tank in quite a different way from the rush of air out of a hole in the tire. And the difference, in a word, is gravity! The main factor behind the flow of water out of the tank is the force of gravity.

For example, for the water tank, it makes a difference to the flow rate whether the hole is at the bottom of the tank or half-way up. But for the tire, the force of gravity is negligible—it doesn't matter whether the hole is at the top or the bottom of the tire—the flow rate will be the same.

The students have a tendency to think of the tire as a balloon, with the air being pushed out because of the elastic force of the tube as it contracts. But when the tube is imprisoned inside the tire, it doesn't stretch as it gets filled. If it did, it would be weaker. So it's not like a balloon at all. In fact it's better to think of the tire as a rigid structure made of hard plastic with air pressure inside.

Let's get down to basics. The air in the tire consists of a large number of molecules, which are constantly in motion. Now what happens when one of those molecules hits the inside surface of the tire? Well, it bounces off. And in fact that's what causes the pressure in the tire. When you poke the tire and push it in, why does it pop back out when you take your finger off?—because all those molecules are colliding with the inside surface of the tire and pushing it back.

Now. What happens if there's a hole so that a molecule heading for the inside surface of the tire "hits" the hole instead? Well it shoots out and escapes. And that's what causes the flow out of the hole. Surprisingly enough, that's all there is to it. Every molecule that flows out is one that was heading for the surface of the tire, minding its own molecular business, and found a hole instead.

Put your finger just above the hole and feel that little jet of air. Those molecules are just escaping by chance? Yes. Wow.

Just to emphasize the difference, suppose we took the water tank out into space where there was no gravity. Then there'd be nothing to stop the water molecules from wandering around the inside of the tank and we'd have to put a lid on the tank or they'd wander out the top. But what about the hole? Well they'd wander out of that too, in just the same way that the air molecules wandered out of the hole in the tire. And in this case, the *z*-curve of the amount of water remaining in the tank wouldn't be a parabola anymore, it would be the same kind of curve as we get for the tire.

So what is the air pressure curve anyway?

It's time to ask just what the tire curve might be. To help you find the answer, I ask the following particular question. Suppose your tire has a small leak. At one point you measure the pressure to be 400 kPa. Suppose over the next minute it drops to 384 kPa —a loss of 16 kPa. Now suppose you leave it for a while until it's dropped to 200 kPa. Half of what it started with. So here's the question—

How much will it lose in one minute now?

Well, here's a simple argument. We need to think in terms of our model—the reason that molecules escape from the hole is that they happen to encounter it in their random motion. When the pressure is 200, there's half as many molecules in the tire as there were when it was 400, so there will be half as many "collisions" with the hole, so the flow rate out should be cut in half. So instead of losing 16 kPa in the next minute, it loses 8.

And so forth. When it has dropped to 100 kPa, it will lose 4 kPa in the next minute. What this argument is really saying is that the amount lost in a minute ought to be always in proportion to the amount in the tire at the start of the minute.

A formula for P. Well, that's a beautifully simple argument, and we can even get an equation out of it. Let's first formulate this air loss principle precisely. What have we said? That the pressure *loss* over a one minute interval is proportional to the pressure at the start of the minute. Alternatively stated:

every minute the tire pressure drops by a fixed percentage.

What is this percentage for our hypothetical example? Well, when $P = 400$, the loss is 16, and when $P = 200$, the loss is 8, and when $P = 100$, the loss is 4, so the 1-minute loss is always 4% of the starting pressure. Thus:

every minute the tire pressure drops by 4%.

Does this give us a formula? Yes it does. Think in terms of the multiplier. A 4% loss leave us with 96% of the amount; thus every minute the pressure is multiplied by 0.96. If the pressure starts at 400, then after t minutes it will be:

$$P(t) = 400(0.96)^t.$$

We have our P-function! Our pressure loss curve is not a parabola—it's an exponential decay function.

I throw this problem out to the class and am impressed by how many of them get it right—and even seem to have a good feeling for the reason. That's the nice thing about molecules—they're quite intuitive.

Parabola and exponential

An interesting "take" on the difference between these two modes of decrease is to ask how the flow rate out of the hole depends on the amount in the vessel. For the tire (exponential decay), as we now see, the flow rate out is proportional to the pressure itself. What about the water tank (parabola)? In this case the flow rate out is proportional to the square root of the water depth.

In the first case, the flow rate will be cut in half when the pressure is halved. In the second case, to cut the flow rate in half we have to cut the amount of water to ¼.

Experiment.

The question we are studying is how the pressure *P* in your tire goes down if there's a small leak, and in the last section we developed an exponential decay model. It's time to perform an experiment to check this out.

In fact it's not so easy to work with a bicycle tire. The problem is that the very act of measuring the pressure alters its value (hiss!) and for something as small as a bicycle tire this can be significant. So we used an old car tire instead which we got from a garage. We pumped it up to 400 kPa (that's a very hard car tire—they are normally just over 200) and one of the students drilled a hole into the side with a small drill, and it worked perfectly. With a large tire, the amount of air lost in the act of measuring the pressure is negligible relative to the amount in the tire. In fact the pump we used had an in-line gauge and all we had to do was monitor that. We used the tire in another class a couple of days later and the hole had to be poked, but it worked fine again. The tire now sits in a safe place in the school and is a resource for all time. Every school should have an old car tire.

We took readings every 5 minutes for an hour. The data are recorded below and plotted at the right. The gauge was such that one could take a reading accurate to 5 kPa.

Who wants the drill?
I held the electric drill up in the air and asked who wanted to drill the hole. I had expected a great rush to the front but there was stillness.

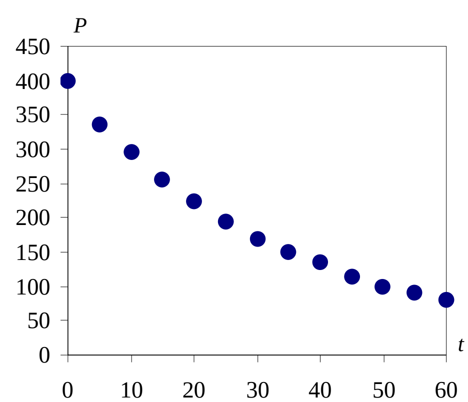
Hasn't everybody always wanted to drill a hole in a car tire? Perhaps they were just awed by the prospect.

Of course after a few moments there was a gratifying response.

time *t* (min)	pressure *P* (kPa)
0	400
5	335
10	295
15	255
20	225
25	195
30	170
35	150
40	135
45	115
50	100
55	90
60	80

This is the data set I got at KCVI the very first time I did this experiment. Needless to say I was a bit nervous. Was it really going to work out the way the theory predicted?

By the way, the tire I used still sits at KC and is brought out every year. Every high school should have an old tire in the basement.



The graph certainly looks reasonable. It has the concave-up shape that our theoretical model predicted.

Now what do we want to do with it? We want to test whether it has the exponential form.

Testing the exponential model

t	P	$\log P$
0	400	2.60
5	335	2.53
10	295	2.47
15	255	2.41
20	225	2.35
25	195	2.29
30	170	2.23
35	150	2.18
40	135	2.13
45	115	2.06
50	100	2.00
55	90	1.95
60	80	1.90

Now we test whether our data has the exponential form:

$$P = Ar^t.$$

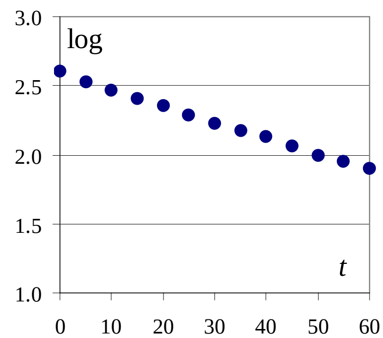
So how do we do that, and how do we find the right values of A and r?

We observed in the last section that the data will have this form if its logarithm plots as a straight line. Just to check that out, hit the above equation with log:

$$\log P = \log(Ar^t) = \log A + t\log r$$

and this is definitely linear in t.

Okay—let's plot the sequence. Holy cow—the points do lie in quite a wonderful straight line.



We conclude that the P-t data is exponential!

We fit a trend-line and get the equation:

$$\log P = 2.5854 - 0.0116t$$

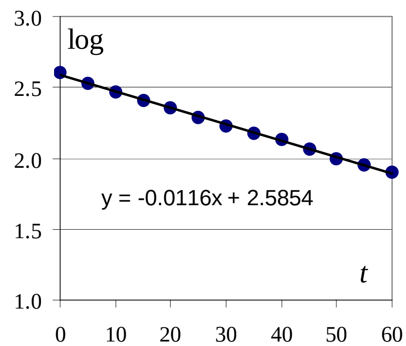
How do we get the P-equation from that? We have to annihilate the log in front of the P and we do that by hitting it with the inverse function 10^x.

$$10^{\log P} = 10^{2.5854-0.0116t}$$

$$P = 10^{2.5854}\, 10^{-0.0116t}$$

$$= 10^{2.5854}\, (10^{-0.0116})^t$$

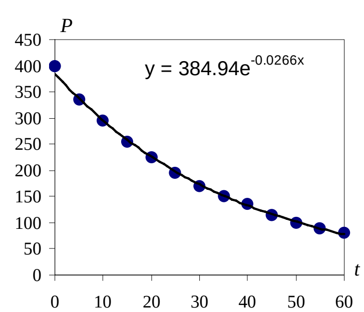
$$= 385\, (0.974)^t$$



Note that the starting value (t=0) that the formula gives us is p=385. That suggests that at the beginning we didn't get the tire quite up to 400.

And what is the percentage loss that our formula predicts? In each minute the pressure is multiplied by 0.974—a 2.6% loss per minute.

Note: Some of the students didn't bother with the logarithm. They simply took the original P data and let technology summon forth an exponential trend line. What they got appears at the right. It looks like a good fit, so the data must be exponential. And they got the equation right away too (except they had to cope with e), and without any work! Ah the wonders of modern technology…



But what did they miss? They missed that wonderful straight line. And that's what I always watch out for because that's the only way I can tell whether my data is really exponential. *We have a way of instantly recognizing when a line is straight. We can't say the same thing about a parabola or an exponential curve. Why not?*

Problems

1. At the moment (t=0) my tire has pressure P = 300 kPa, but it has a slow leak and loses 5% of its pressure every hour.
(a) What will its pressure be after ten hours (t=10)? [Answer: 179.6.]
(b) Find a formula for P after t hours. [Answer: $P = 300(0.95)^t$.]
(c) At what time will P = 150 kPa? [Answer: t=13.5 h.]

2. At the moment (t=0) my tire has pressure 360 kPa, but it has a slow leak and loses 2% of its pressure every hour.
(a) What will its pressure be after one hour (t=1)?
(b) What will its pressure be after five hours (t=5)?
(c) Find a formula for its pressure P after t hours.
(d) What is the percentage loss in pressure over a ten hour period?
(e) How long until the pressure drops to 10 kPa?

3. My bike tire has pressure 360 kPa, but after a week this has fallen to 320 kPa.
(a) What will it be after another week?
(b) Find a formula for the pressure P after t weeks.
(c) I can't be bothered to fix the tire and I refuse to ride it when it's below 100 kPa and I don't have a pump and I can only go to the gas station once a week on Saturday mornings, and you have to pay 25 cents for air, so I only fill it up when I have to. Assuming that each pump-up brings it up to 360 kPa, how long is the interval between pump-ups?

4. The data on the left was collected from a leaking tire. Plot $\log P$ against t and determine that this data lies in a reasonable straight line. Draw the best-fit line, either with eyeball and ruler or with a regression routine, and find its equation and transform this equation into an exponential equation for P.

t	P
0	320
5	290
10	268
15	250
20	235
25	219
30	202
35	190
40	178
45	169
50	157
55	149
60	138
65	130
70	122
75	115

5. I measure my car tire pressure at noon, at 1 PM and at 2 PM. During the first hour it lost 30 kPa and during the second hour it lost 26 kPa. What was the pressure at noon?

6. I measure my bike tire pressure at noon, at 1 PM at 2 PM and at 3 PM. During the second hour it lost 80% of what it lost during the first hour and during the third hour it lost 50 kPa. What was the pressure at noon?

Relative and absolute pressure revisited

So why is a 500 kPa tire in a 100kPa space the same as a 400 kPa tire in a 0kPa space? The answer is that air goes out of the 500 kPa tire a bit faster than the air goes out of the 400 kPa tire. But when the tire is sitting in a 100 kPa environment, *there's also air going in*. That seems strange at first, that air goes into the hole at the same time as air rushes out, but that's what happens. When you put the 500 kPa tire in the 100 kPa space, the air goes out at 5 times the rate that the air goes in. When you subtract the air going in from the air going out, the resulting air loss is exactly the same as it would be for a 400 kPa tire in an empty space.