2.3 Fitting the tide

Find a vertical pier at the edge of the sea, put a scale down the side, and measure the height of the water against time and you’ll get a graph something like that below—data taken in the Bay of Fundy over a 4-day period. This region is famous for its unusually high tides, a resonance effect caused by the funneling shape of the basin. As you can see from the graph, the amplitude of these tides is almost 6 meters whereas on the ocean, tides are a third or a quarter of that.

![Graph of tides](image)

*The tides are often cited as an example of periodic behaviour. The period of these oscillations is the time to complete one full cycle, and in the above graph appears to be just over 12 hours. In fact a theoretical calculation, using the period of the moon (see Problem 9), shows the period to be close to 12 hours and 25 minutes which is just about 12.42 hours. Thus in a 24-hour period there are just under two high tides and two low tides.*

The problem is to construct a sinusoidal function which “fits” the above graph. Just to give you some simple numbers to work with, suppose the graph oscillates between height \( y = 1.5 \) and \( y = 11.5 \). To calculate the period, suppose it crosses the mid-line \( (y = 6.5) \) at times \( t = 0 \) and \( t = 92 \). Find a sine function

\[
y = H(t) = a \sin(bt+c) + d
\]

which has these properties.

**Solution**

A good strategy is to start with the standard sin graph and then work towards our target picture bit by bit.

Okay: Here’s the graph:

\[
H_0(t) = \sin t.
\]

I have called this \( H_0 \) because it’s the starting candidate. Now there are lots of things wrong with \( H_0 \) and we’re going to fix them one at a time. That will take us through a sequence of functions of \( t \) each getting closer to the \( H(t) \) function we want, and to keep track of them we’ll give them different names, \( H_0, H_1, \) etc. There’s a judgment call to be made as to the order in which things should be fixed up. What I’ve done here is not the only possibility, but some are definitely more awkward than others.

*In a sense, this is a game the class can play—decide which problem to fix next, and how to do it.*
Getting the period right.

Let’s fix the period first. The sin function has period $2\pi$ (which is just over 6), but our tide function has 7.5 periods in 92 hours (that was given at the beginning) so has period $92/7.5 \approx 12.27$ so we have to expand the $t$-axis by a factor of close to 2—more precisely, by a factor of $12.27/2\pi$. Now how do we do that algebraically?

Well I find I have to think carefully to “get that right.” If we call our new function $H_t(t)$, then here’s what I say to myself: “What $H_0(t)$ does as $t$ goes from 0 to $2\pi$ is the same as what $H_t(t)$ has to do as $t$ goes from 0 to 12.27. In particular:

$$H_t(12.27) = H_0(2\pi)$$

And other values work in proportion:

$$H_t(12.27r) = H_0(2\pi r)$$

Now to find the value of $H_t$ at any $t$, substitute $t = 12.27r$ on the left, and then $r = t/12.27$ on the right. Thus:

$$H_t(t) = H_0\left(\frac{2\pi}{12.27}\right).$$

Getting the amplitude right.

Now let’s adjust the amplitude. The vertical spread of the sin function is 2 (from $-1$ to $+1$) but the spread of our tidal function is 10 (from 1.5 to 11.5). So we need to multiply by the ratio of these two numbers which is 5. Our new approximation is

$$H_2(t) = 5 \sin\left(\frac{2\pi}{12.27}\right).$$

Getting the height right.

Now we adjust the vertical position. The current function goes from $-5$ to $+5$, and we want to go from 1.5 to 11.5, so we must lift it by 6.5. This gives us

$$H_3(t) = 5 \sin\left(\frac{2\pi}{12.27}\right) + 6.5.$$
Getting the phase right.
What we have now is really a pretty good version of our target graph. Of course the two graphs “start off” in a different way. That’s a question of what’s called the “phase” of the oscillation—what you choose as the time origin. If we wanted the graph of \( H_3 \) to start in the right way, we’d have to shift it to the left by half a period, that’s \( 12.27/2 = 6.135 \).

How do we do that algebraically? [Horizontal shifting always requires care—it’s so easy to get the sign wrong.] What we want (look at the picture!) is for the value of the new function \( H_4 \) at any \( t \) (e.g. \( t=0 \)) to be the same as the value of \( H_3 \) at \( t+6.135 \):

\[
H_4(t) = H_3(t+6.135)
\]

This gives us our function

\[
H_4(t) = 5 \sin \left( \frac{2\pi(t + 6.135)}{12.27} \right) + 6.5
\]

and this simplifies to

\[
H_4(t) = 5 \sin \left( \frac{2\pi t}{12.27} + \pi \right) + 6.5
\]

This is plotted at the right.

Finally, the two graphs are plotted together below, the Joggins Wharf data as a set of points and \( H_4 \) as a curve. It’s a rather nice fit.

Note added. One of my students remarked that there was an easier way to get the phase right. The only problem with the \( H_3 \) graph (at the top of the page) is that it goes up where the real graph goes down. So all we have to do is reverse its sign. Well, not quite. The trouble is that we lifted it already. Go back to \( H_3 \) which is centred at \( y=0 \). Change its sign, giving us

\[
H_3(t) = -5 \sin \left( \frac{2\pi t}{12.27} \right)
\]

(plotted at the right) and then lift this by 6.5 and we’re home! We get:

\[
H_4(t) = -5 \sin \left( \frac{2\pi t}{12.27} \right) + 6.5.
\]

This has a different algebraic form from our first \( H_4 \) expression, but they are the same function.
Problems

1. Find the equation of a sinusoidal oscillation which has period 10, oscillates between \( y = 6 \) and \( y = 20 \), and takes the minimum value \( y = 6 \) at \( t = 0 \). [One version of this is \( y = 7\cos(2\pi t/10 - \pi) + 13 \).]

2. At the right is plotted some periodic data. Find a sinusoidal function (a sine or a cosine) which should give a good fit to the data.

3. What is the period of the following functions? That is, what is the minimum value of \( p \) for which \( f(t) = f(t+p) \) for all \( t \)?

   (a) \( 4\sin(\pi t/4) \)
   
   (b) \( \cos(1+2\pi t) \)
   
   (c) \( \sin(\pi t/4) + 4\sin(\pi t/2) \)
   
   (d) \( \frac{2 + \sin t}{2 + \cos t} \)
   
   (e) \( \frac{\sin t}{\cos t} \)

4. Make a sketch of the following graphs that shows at least two full periods.

   (a) \( y = 2\cos(2\pi t/5) + 4 \)
   
   (b) \( y = 6 - 4\sin(2\pi t/4) + 4 \)

5. At the right is plotted some periodic data. Find a sinusoidal function (a sine or a cosine) which should give a good fit to the data.

6. A pole stands vertically in the middle of a flat plain somewhere north of the arctic circle in mid-summer. [So that the sun never sets.] Draw a graph of the length of its shadow against time. Is this a sinusoidal oscillation? Can you find the form of the equation? [This is probably quite a nice problem.]
7. There are lots of periodic functions around which go up and down in a smooth way and look a lot like sine curves. One interesting example is the amount of sunlight (i.e. the number of hours each day that the sun is above the horizon) as a function of time throughout the year. Interestingly enough, even with all possible simplifying assumptions such a circular orbit of the earth around the sun, a spherical earth etc., it’s still not exactly sinusoidal. But it is in fact very close to a sine curve and in practice the sine curve can be used to make accurate predictions.

The graph was plotted from data in which the maximum # daylight hours was 15:44 obtained on June 21, and the minimum was 8:40. Find a sinusoidal equation (using either the sine or the cosine) for the graph, measuring \( t \) in units of days, and taking \( t=0 \) to be on June 21. Assume that this is not a leap year. Use your equation to predict the number of daylight hours on July 21, September 21, and December 21. The data give 15:11, 12:14 and 8:41, respectively.

8. The period of the tides.
The period of the tides is the time between successive high tides. Now the movement of the tides is caused by the fact that the earth revolves on its axis, and since it performs one revolution in every 24-hour period, the diagram at the right tells us that we should get one high tide every 12 hours. So the period is 12 hours. Right?

Wrong! The difficulty is that not only the earth revolving on its axis, but the moon is rotating around the earth (with a period of close to 30 days). So if we were to start at high tide—the one on the moon side—then after 24 hours when the earth has turned a full revolution, we won’t be quite at the peak tide again because the moon will have moved, so that the peak tide will have shifted by about 1/30 of a revolution, that is, by about 12°. So to get to the peak we have to travel some more. Or is it less? Which way does the moon move anyway? Well it moves the same way that the earth rotates. So by the time the earth has completed a full rotation, the peak tide has moved up by 12°. So to get there, the earth has to rotate an additional 12° and that will take an extra 1/30 of a day.

But wait a minute. By the time the earth gets there, the moon will have moved a bit more…. Hmm. How do we work this one out anyway. How do we calculate the period of the tides?

Here’s one way to proceed. Let \( \theta \) be the extra angle through which the earth has to rotate to see the moon directly overhead again. For convenience measure \( \theta \) in units of “revolutions.” Thus 360° is 1 revolution and \( 1+\theta \) is the angle the earth has to rotate through to complete two tidal periods. Now in that time, the moon travels a fraction \( \theta \) of its period around the earth. By putting these two bits of information together, we should be able to calculate the exact period of the tides. By the way, you’ll need the exact period of the moon around the earth which is 29 days, 12 hr, 44 min, and 2.8 sec. [The answer is that the period of the tides is 12 h, 25 min and 14 s.]