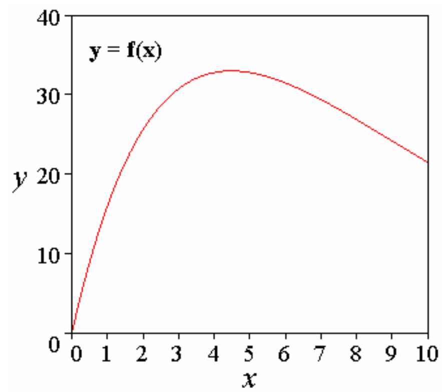
5.1 Average and instantaneous rate of change

Average rate of change

At the right is a graph of a function *f*. We can think of the function in many ways, but for now I'm going to think of the horizontal axis as time (though I will call it *x* rather than *t*) and then *f*(*x*) will represent the size of something changing over time. It could be the size of a growing (and shrinking population), the position of a walker walking along a road, or the temperature of an object sitting outside on a summer day. I will go with the last one: *f*(*x*) *will be the temperature of a mug of tea placed on the picnic table over the course of a 10-hour day.*

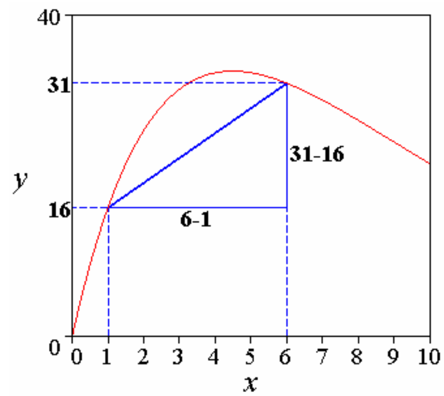


We see that the temperature starts at the freezing point increases quite rapidly for the first couple of hours, then its rate of increase slows down and after about 5 hours starts to decrease.

Our purpose here is to look at average rates of temperature change and to interpret these on the graph.

For example, over the 5 hour interval [1, 6], the temperature increases from 16° to 31°, a 15° net increase. We say that the temperature changes over that interval at the average rate of 15/5 = 3°/h. More formally, the average rate of temperature change is calculated as:

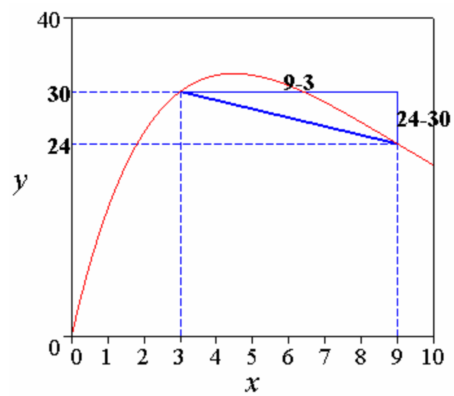
$$\frac{\Delta y}{\Delta x} = \frac{\text{change in } y}{\text{change in } x} = \frac{31-16}{6-1} = \frac{15}{5} = 3\,.$$



By the way, note well how that quotient 15/5 is represented in the diagram. Since it's a change in *y* divided by the corresponding change in *x*, it's the slope of a chord drawn on the graph, called a *secant.*

Similarly, on the 6-hour interval [3, 9], the temperature decreases from 30° to 24°, a net 6° decrease. We say that the temperature changes over that interval at the average rate of –6/6 °/h, or minus one degree per hour.

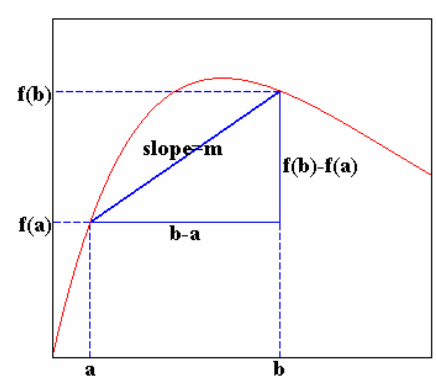
$$\frac{\Delta y}{\Delta x} = \frac{24-30}{9-3} = \frac{-6}{6} = -1\,.$$



Again, this is the slope of a secant, except this time it has negative slope.

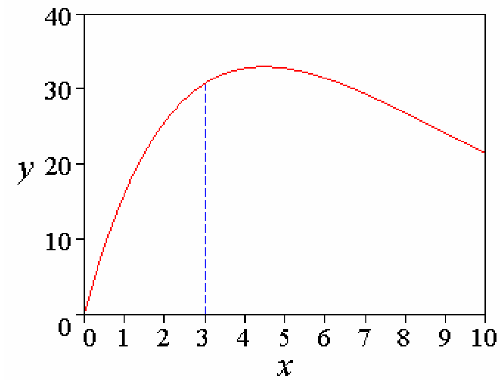
Here's the formal definition: *the average rate of change of f*(*x*) *on the interval* $a \le x \le b$ is defined as

$$m = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}\,.$$



Geometrically, this is the slope of the secant drawn to the graph over the interval [*a*, *b*].

Instantaneous rate of change.
I ask my students what the rate of increase of temperature might be *exactly* at time $x = 3$. We call that the instantaneous rate of change. How might we calculate it? That engenders an interesting discussion.

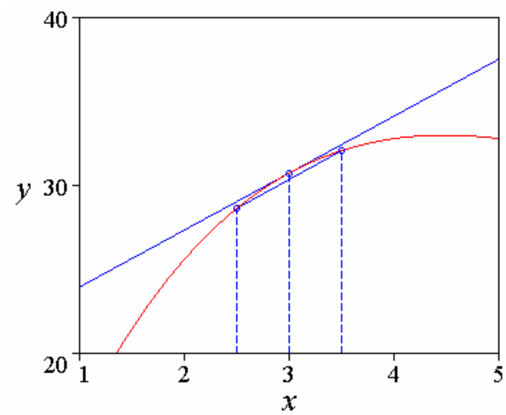


In a sense what we want is the average rate of change on the interval [3, 3]. Algebraically, that would be:

$$\frac{\Delta y}{\Delta x} = \frac{f(3) - f(3)}{3 - 3}$$

but that gives us 0/0—not something we can calculate. And geometrically it would be a secant to the graph drawn from one point to the *same* point, and that doesn't give us a line segment at all.

A number of ideas are put forward. The most popular seems to be to take a tiny interval whose midpoint is 3 and take the slope of the secant on that. Others say we should take the slope of the tangent at x=3. Some students claim that that those would give you the same thing.

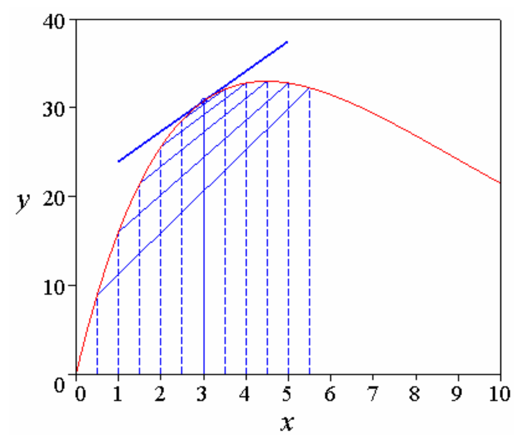


For example, at the right (in a blown-up version of the graph) I have drawn the secant on the interval [2.5, 3.5] which is the 1-hour interval centred at x=3. I have also drawn the tangent to the curve at x=3. These certainly seem to have slopes which are almost the same. And if we took an interval that was smaller, for example from 1 minute before 3 to 1 minute after 3, and measured the average rate of change on that, we'd surely get a measure that was in practice indistinguishable from the instantaneous rate at $x = 3$.

In fact that's more or less how the general theory goes. Formally, the *instantaneous rate of change of* $f(x)$ *at* $x = a$ is defined to be the limit of average rates of change on a sequence of shorter and shorter intervals centred at x=a. Since an interval centred at x=a always has the form [a–h, a+h] (with length 2h), this can be written:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

Geometrically, it is the slope of the tangent to the graph of f at $x = a$.



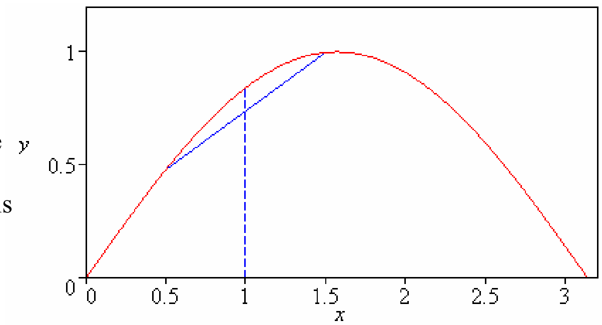
It's neat how everything works here. If we have a sequence of secants, centred at x=a, getting shorter and shorter, their slopes ought to approach the slope of the tangent at x=a. But since their slopes are average rates of change on increasingly short intervals, they ought also approach the instantaneous rate of change at x=a. So that makes the slope of the tangent equal to the instantaneous rate of change!

Example 1. By taking a sequence of increasingly short secants centred at x=1, estimate the slope of the tangent to the curve
y =sin$x$ at x=1.

Solution.
A typical secant is drawn at the right. It has slope:

$$\frac{\Delta y}{\Delta x} = \frac{\sin(1+h) - \sin(1-h)}{2h}$$

for the value h = 0.5. To get good accuracy, I use a sequence of h values smaller than that. In fact I start with h=0.1, and reduce each successive h by a factor of 10. The table below is generated with my ***hp*** calculator but I assume that a standard TI would give comparable results. The display gives me 9 decimal places, but internally the machine seems to carry a couple more places, perhaps 11. Take some time to study the table carefully and try to interpret what's going on. What do you notice?



h	sin(1–h)	sin(1+h)	Δy = sin(1+h)–sin(1–h)	slope = Δy/2h
0.1	0.783326910	0.891207360	0.107880450	0.539402253
0.01	0.836025979	0.846831845	0.010805866	0.540293300
0.001	0.840930262	0.842010866	0.001080604	0.540302200
0.0001	0.841416950	0.841525011	0.000108060	0.540302000
0.00001	0.841465582	0.841476388	0.000010806	0.540305000
0.000001	0.841471525	0.841470445	0.000001081	0.540300000

As the secants get shorter we expect to get closer and closer to the value of the tangent at x=1. But is that what the last column is giving us? *No it is not*, because something else is happening that works *against* increased accuracy and that the loss of decimal places. The values of Δy (second-last column) that the calculator produces are accurate to the internal accuracy of the machine (say 11 places) but when we divide by 2h the decimal point moves a number of places to the right and we pick up a few meaningless zeros at the end of Δy.

I ask the class which of the values in the last column they think might be closest to the true value of the slope of the tangent. I take a vote and the third row wins: h = 0.001 giving an estimate of

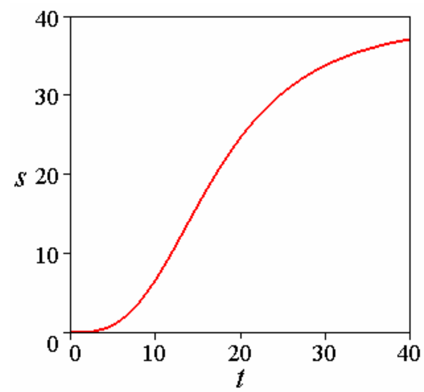
$$m = 0.540302200.$$

It's interesting to compare the calculator results above with those generated by Excel, each entry again displayed to 9 decimal places. The recorded values of Δy are essentially the same, but we get much better quotients in the last column. Excel clearly carries "enough" places for these calculations. It appears as if the class voted correctly!

h	sin(1–h)	sin(1+h)	Δy = sin(1+h)–sin(1–h)	slope = Δy/2h
0.1	0.783326910	0.891207360	0.107880450	0.539402252
0.01	0.836025979	0.846831845	0.010805866	0.540293301
0.001	0.840930262	0.842010866	0.001080604	0.540302216
0.0001	0.841416950	0.841525011	0.000108060	0.540302305
0.00001	0.841465582	0.841476388	0.000010806	0.540302306
0.000001	0.841471525	0.841470445	0.000001081	0.540302306

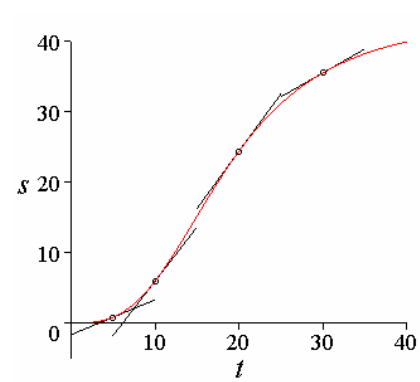
In fact calculus is able to produce exact answers for tangent slopes. The slope of sin$x$ at x=a turns out to be cos$a$. Thus the slope at x=1 is cos(1) = 0.540302306. This is exactly what the last two rows of the Excel table give us.

Example 2. I walk along a straight path for 40-seconds covering a distance of exactly 40 meters. The graph at the right displays the distance *s* that I walk (m) against time *t* (s). The following questions should be answered with the help of a construction on the graph.



(a) Describe how my velocity changes over the course of the 40-second period.
(b) At what time is my velocity a maximum and what is that maximum value?
(c) What is my average velocity over the first 20 seconds?
(d) At what time *t* is my average velocity over the first *t* seconds a maximum and what is that maximum value?
(e) A friend standing beside me at time *t*=0 walks along the same path during the same period but at a constant speed of 1 m/s. When are we the farthest apart?
(f) Another friend walks the same path during the same 40-second interval at constant speed of 1 m/s but in the reverse direction. At what point on my path do we meet one another?

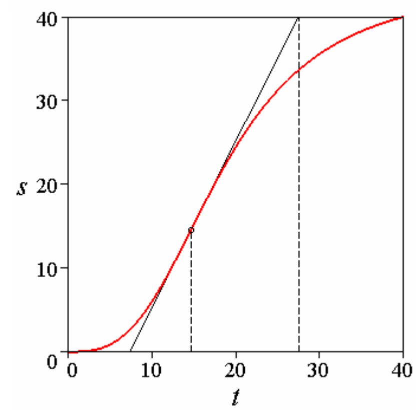
Solution
(a) My velocity at any point is the slope of the tangent to the graph. It is clear from the way the graph curves that the slope increases during the first part of the walk and decreases during the second part. Thus I accelerate during the first part and decelerate during the second part.



(b) My velocity is a maximum at the point at which the slope stops increasing and starts decreasing. This is a bit hard to estimate accurately from the graph (so there will be considerable variation in student answers) but seems to be somewhere between *t*=10 and *t*=20, say at *t*=15. We draw the tangent at this point and measure its slope to be approximately:

$$\frac{\text{rise}}{\text{run}} = \frac{40}{27-7} = \frac{40}{20} = 2$$

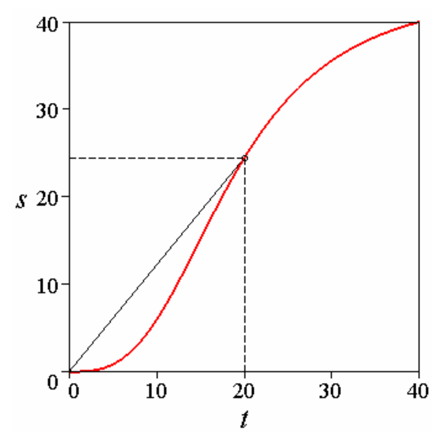
My maximum velocity is approximately 2 m/s.



(c) My average velocity over the first 20 seconds is the slope of the secant from the origin to the point on the graph at *x*=20. This is measured to be approximately

$$\frac{\text{rise}}{\text{run}} = \frac{24}{20} = 1.2$$

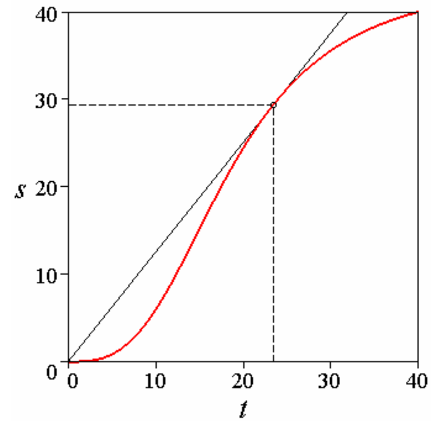
My average velocity over the first 20 seconds is 1.2 m/s.



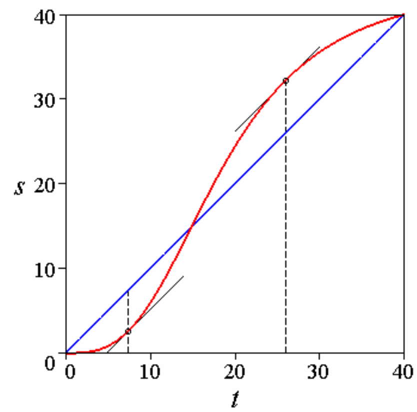
(d) My average velocity over the first *t* seconds is a maximum where the slope of the secant from the origin to a point on the graph is a maximum and that occurs when the secant is tangent to the graph. This is measured to be approximately at *t*=23, at which time I have traveled approximately 29 m, for an average velocity of

$$\frac{\text{rise}}{\text{run}} = \frac{29}{23} = 1.26\ .$$

My maximum average velocity is 1.26 m/s

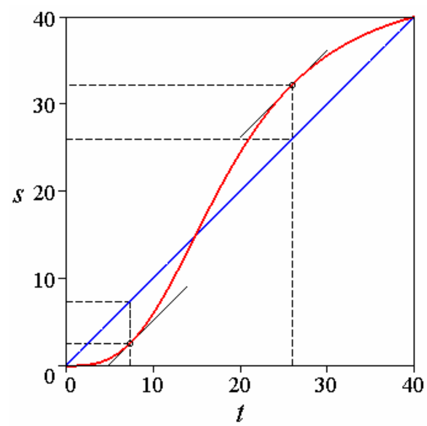


(e) My friend's graph is a straight line of slope 1. The distance between me and my friend at any time is the vertical distance between the graphs at that time. The key observation is that for this to be a maximum, *my friend and I must have the same velocity*. [Making this argument rigorous is a significant and non-trivial task which is left for the calculus course. But it's important to engage the intuitive argument here. One idea is to focus on the distance between me and my friend. If our velocities are different, this changes at a non-zero rate and can't be at a maximum.]

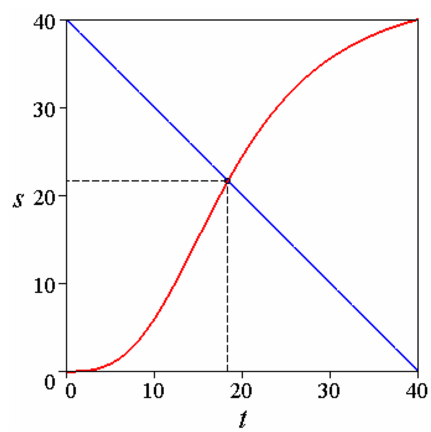


There are two places at which my graph has slope 1: at approximately *t*=7 and *t*=26. At the first of these my friend is ahead`of me and at the second, she is behind. The two distances between us look to be very similar. Which is the greater?

This is easier to assess by drawing horizontal lines. The second time seems to have the greater distance. The distance between us is a maximum at approximately *t*=26.



(f) The graph of my other friend is a straight line of slope –1. We meet at approximately *t*=18 at which point I have gone a distance of approximately *s*=22 m.



Example 3. In estimating slopes of tangents at a point x, we have been using secants on intervals centred at x. Are there any functions for which this estimation is always exact? Well, of course this would be the case for linear functions as these have a constant slope m and all secants also have slope m. But are there other functions? Yes there are. *Parabolas have this property*. Your job is to demonstrate that.

(a) Take the quadratic polynomial $f(x) = x(3-x)$. Show that all secants centred at $x=2$ have the same slope.
(b) Now take general quadratic polynomial:

$$f(x) = ax^2 + bx + c.$$

Show that all secants centred at a fixed value of x have the same slope.

Solution.
(a) Secants centred at $x=2$ sit on an interval $[2-h, 2+h]$ for some $h>0$ and have slope

$$\frac{f(2+h)-f(2-h)}{2h}$$

Our job is to show that this is independent of h.

$$f(2+h) = (2+h)(3-2-h) = (2+h)(1-h) = 2-h-h^2$$
$$f(2-h) = (2-h)(3-2+h) = (2-h)(1+h) = 2+h-h^2$$

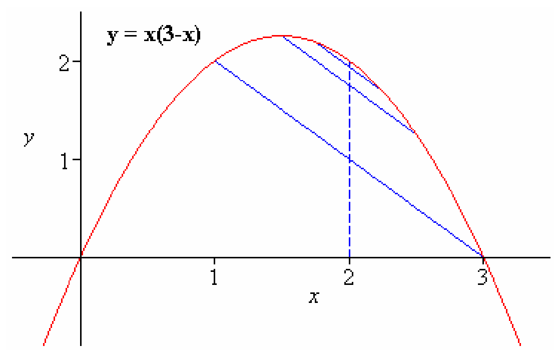
Now subtract:

$$f(2+h) - f(2-h) = (2-h-h^2) - (2+h-h^2) = -2h.$$

The slope of the secant is

$$\frac{f(2+h)-f(2-h)}{2h} = \frac{-2h}{2h} = -1$$

and this is indeed independent of h.



A parabola has the property that the family of secants centred at a fixed point all have the same slope. This of course will also be the slope of the tangent at that point. In the diagram above, a number of secants centred at $x=2$ are drawn. They are seen to be parallel.

(b) Now we work with the general quadratic polynomial:

$$f(x) = ax^2 + bx + c$$

and we need to show that:

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x-h)}{2h}$$

is independent of h.

$$f(x+h) = a(x+h)^2 + b(x+h) + c = \boxed{a(x^2+2xh+h^2)+b(x+h)+c}$$

$$f(x-h) = a(x-h)^2 + b(x-h) + c = \boxed{a(x^2-2xh+h^2)+b(x-h)+c}$$

Now subtract:

$$f(x+h) - f(x-h) = \boxed{a(4xh)+b(2h)}$$

Thus:

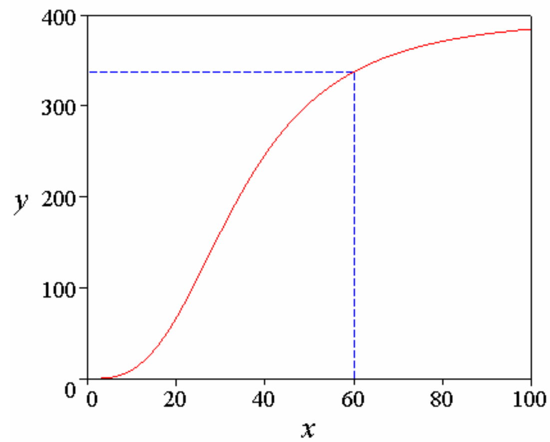
$$\frac{\Delta y}{\Delta x} = \frac{4axh+2bh}{2h} = 2ax+b$$

And this is indeed independent of h. So all secants centred at x have the same slope.

Of course if all secants centred at x have the same slope, this will be the slope of the tangent at x. Thus, we have discovered a formula for the slope of the tangent to a quadratic polynomial at any point. The slope of $y = ax^2 + bx + c$ at any x is:

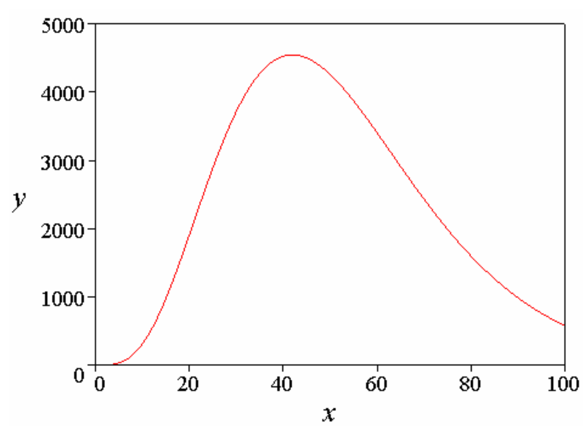
$$\frac{dy}{dx} = 2ax + b.$$

Problems.



1. You are trying to sell "eco-log" T-shirts to support a local conservation project. To sell any at all you will need to some advertising and in fact the graph at the right depicts the total number *y* of T-shirts you can expect to sell against the total amount *x* you spend on advertising. For example, if you buy $60 worth of advertising, you can expect to sell 340 T-shirts. Where appropriate, answer the following questions with a construction on the graph.

(a) The slope of the graph at any point is called the marginal return at that *x*. Explain exactly what that means in language that your sister in grade 10 would be able to understand.
(b) I might have expected that the marginal return would be greatest at the beginning, but this is not the case. Explain why this seems reasonable.
(c) At what value of *x* is the marginal return the greatest?
(d) What size of investment in advertising has the property that the average number of T-shirts sold per dollar spent is the greatest?
(e) Measure the slope of the secant on [40, 60]. Interpret this quantity in simple language. [Hint: suppose you have decided to spend $40 on advertising but now you are having second thoughts and are contemplating spending an additional $20…]



2. During the course of an influenza epidemic, the number of infected individuals (called the *size* of the epidemic) increases quite quickly attaining a maximum (the *peak* of the epidemic) and then decreases as people recover and the number of new cases tails off. The graph at the right depicts the size of a recent epidemic over a 100 day period. Answer the following questions with constructions on a copy of the graph. Show your construction.
(a) Find a 40-day period over which the average rate of increase of the epidemic was zero.
(b) Find a 20-day period over which the size of the epidemic increased at an average rate of 50 individuals per day.
(c) At what point was the average growth rate of the epidemic a maximum (measured from time x=0)?
(d) At what point was the instantaneous growth rate of the epidemic a maximum?
(e) Find a point at which the size of the epidemic was decreasing at an instantaneous rate of 100 individuals per day.

3. By taking a sequence of increasingly short secants centred at x=1, estimate the slope of the tangent to the curve $y = 2^x$ at x=1. Use a standard calculator. Use a diagram to give a graphical interpretation of your work.

4. (a) Show that the secant property of Example 3 does *not* hold for cubic polynomials. That is, the family of all secants centred at a particular *x* does *not* have a constant slope.

(b) As a side-result of your calculation, find a formula for the slope of a cubic polynomial at any *x*.